АВТОРСКИЙ УКАЗАТЕЛЬ

ТЕЗИСЫ ДОКЛАДОВ

A NOTE ON QUASIFIELDS

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The investigations of problems of construction and classification of quasifields from 1960 (Kleinfeld E., 1960; Knuth D.E.,1963) usually use computer calculations. The development to 2007 is reflected by N.L. Johnson, V. Jha, M.. In 2014, V.M. Levchuk, S.V. Panov, P.K. Shtukkert gave the structure of finite quasifields. In this paper we discuss on finite quasifields and semifields.

AUTOMATIC CLASSIFICATION BASED ON MULTIDIMENSIONAL INFORMATION REPRESENTATION

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Automatic classification is most often associated with the decision of the question about proximity of a given point to the center pointcloud (the standard) grouped by virtue of the general property and the finding of a given point in a given region [1]. A problem is posed and solved, while the sum of all the coordinates and all the relations between the coordinates is chosen as the general property. Inductive reasoning in two-dimensional and then in three-dimensional spaces proves that the surface on which the sum of coordinates is constant is a plane, respectively in a multidimensional space this will be the hyperplane, or rather the family of hyperplanes depending on the value of the constant. Further, the closeness of the point on this surface to the point of intersection with a given ray originating from the origin is estimated, and by comparison with the given radius the neighborhood of these points is classified. In this case, the ray represents the class of points for which the ratio of all coordinates is the same. Since in the general case the revealed hyperplane is not perpendicular to the ray, the region of proximity to the points forming the ray in the multidimensional space is a cylindrical surface with an elliptical cross section. In addition, by passing through a given ray of the plane perpendicular to the coordinate planes, it is possible to further classify the neighborhood of a given point with each of the coordinate axes as it is found between the corresponding boundaries [2].

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INVERSE DESIGN METHOD IN CLOAKING PROBLEMS

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In recent years much attention has been given to creation of tools of material objects masking from detection with the help of electromagnetic or acoustic location. Beginning with pioneering papers by Pendry et al. [1] the large number of publications was devoted to developing different methods of solving the cloaking problems. Transformation optics (TO) is the most popular method of designing cloaking devices (hereafter, cloaks). In fact, this is the mathematical method that enables us to design suitable material structures (metamaterials) that when placed around the object may drastically suppress its scattering and therefore to hide the object from detection of radar system. The methodology of cloaking based on this method obtained the name of direct design because it is based in fact on solving direct problems of electromagnetic scattering.

It should be noted that this design possesses several drawbacks. In particular some components of spatially dependent parameter tensors of ideal cloak are required to have infinite or zero values at the inner boundary of the cloak which are very difficult to implement. There are several approaches of overcoming these difficulties. The first one consists of replacing "exact" cloaking problem by approximate cloaking problem which solutions admit relatively simple technical realization.

Alternative approach consists of using optimization method for finding solutions of approximate cloaking problems. Based on optimization method the new cloaking design methodology became develop recently. It obtained the name of inverse design since it is related with solving inverse electromagnetic (or acoustic) problems [2]. Using this methodology enables us to take into account different constraints for designing cloaks and in particular to solve some limitations connected with technical realization of solutions of cloaking problems. In this paper we discuss some results which were obtained by the authors while theoretical study of cloaking problems for electromagnetic, acoustic and static physical fields.

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THE ESTIMATION OF GROWTH RATE FOR THE HYPERELLIPTIC SYSTEMS OF SEQUENCES

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Let $A, B : \mathbb{Z} \to \mathbb{C}$ be not-zero sequences such that $\exists\ 2k_0 + 2k_1$ other sequences

$$C_1^{(0)}, \dots, C_{k_0}^{(0)}, D_1^{(0)}, \dots, D_{k_0}^{(0)} : \mathbb{Z} \to \mathbb{C},$$
$$C_1^{(1)}, \dots, C_{k_1}^{(1)}, D_1^{(1)}, \dots, D_{k_1}^{(1)} : \mathbb{Z} \to \mathbb{C},$$

and for each $m, n \in \mathbb{Z}$

$$A(m+n)B(m-n) = \sum_{j=1}^{k_0} C_j^{(0)}(m)D_j^{(0)}(n),$$

$$A(1+m+n)B(m-n) = \sum_{j=1}^{k_1} C_j^{(1)}(m)D_j^{(1)}(n).$$

We shall call such couple sequences (A, B) as *the hyperelliptic system* of 0-rank $k_0 = R_0(A, B)$, 1-rank $k_1 = R_1(A, B)$ and the rank $k = R(A, B) = \max(k_0, k_1)$ ([1]). Here k_0 and k_1 are smallest possible nonnegative integers.

Teorem. *Let (A, B) be a hyperelliptic system of sequences. Then for some real $\Delta = \Delta(A, B) > 0$*

$$|A(m)| + |B(m)| \underset{A,B}{\ll} \exp\left(\Delta m^2\right).$$

This work is supported by the Russian Foundation for Basic Research (project №17-01-00225).

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ON LINEAR FRACTIONAL TRANSFORMATIONS OF THUE POLYNOMIALS

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The paper is concerned with the algebraic aspects of the theory of Thue polynomials.

Let $f(x)$ be a irreducible polynomial of n-th degree with integer coefficients and leading coefficient is equal to 1 and α_ν be a root of this polynomial, let $P(t)$ and $Q(t)$ be the polynomials with integer coefficients. Then $\mathcal{T}(t, \alpha_\nu) = P(t) - \alpha_\nu Q(t)$ is called *Thue polynomial* for α_ν.

An order of Thue polynomial $\mathcal{T}(t, \alpha_\nu)$ is called the highest degree j such that $\mathcal{T}(t, \alpha_\nu)$ is divided by $(t - \alpha_\nu)^j$. We denote Thue polynomial of j order by $\mathcal{T}_j(t, \alpha_\nu)$.

We continue the research carried out by M. N. Dobrovolsky and V. D. Podsypanin in the 60s of the XX century. The principal difference of their approach in the study of Thue polynomials was to find the explicit form of these polynomials. Dirichlet principle allowing to establish the existence of such polynomials with given properties was replaced by recurrence relations for the main Thue polynomials of order j and $j + 1$.

In particular, using a linear fractional transformation, the connection between the Thue polynomial of order j for the number α and the corresponding Thue polynomial for the residual fraction α_m for continued fraction expansion α was established.

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MOMENTS OF L-FUNCTIONS AND THE LIOUVILLE-GREEN METHOD

O. G. Balkanova (University of Turku, Turku)

In this talk, we show that the percentage of primitive cusp forms of level one and weight $4k \to \infty$, $k \in \mathbf{N}$ for which the associated L-function at the central point is no less than $(\log k)^{-2}$ is at least 20% for an individual weight and at least 50% on average. The key ingredients of our proof are the Kuznetsov convolution formula and the Liouville-Green method. This is a joint work with Dmitry Frolenkov.

ON SYMPLECTIZATION OF 1-JET SPACE AND CONTACT BIRATIONAL TRANSFORMATIONS

P. V. Bibikov (Institute of Control Sciences RAS, Moscow)

Consider the contact 1-jet space $J^1\mathbb{k}$ over field $\mathbb{k}$ with canonical coordinates (x, y, p). In this paper we study the birational contact transformations of $J^1\mathbb{k}$, which are affine in coordinates (x, y). The corresponding group of such transformations is called *affine p-group.* This group is interesting in connection with the so-called Klein–Keller conjecture: *the group of contact birational transformations os generated by birational point transformations and by Legendre transformations.*

The polynomial version of this conjecture was proved by M. Gizatullin in [1]. Also contact birational transformations were studied in many other works (see, for example, [2]).

In this work we suggest a new approach to study this question. In standard way the main instrument for such problems is the algebraic geometry and geometry of Cremona groups. But we apply differential geometric point of view to study this problem. Namely, we consider the so-called symplectization $\mathrm{Symp}(J^1\mathbb{k})$ of contact space $J^1\mathbb{k}$ and lift the birational contact transformations of $J^1\mathbb{k}$ to the birational symplectic transformations of $\mathrm{Symp}(J^1\mathbb{k})$.

Symplectization $\mathrm{Symp}(J^1\mathbb{k})$ makes it possible to describe the structure of the affine p-group.

Theorem 1. *Each element of the affine p-group can be represented as $L \circ \Phi \circ L$, where L is the Legendre transformation and Φ is the point transformation*

$$x \mapsto \frac{ax + b}{cx + d}, \quad y \mapsto A(x)y + B(x)$$

(here A, $B \in \mathbb{k}(x)$ are arbitrary rational functions).

Theorem 2. *Let field $\mathbb{k}$ be algebraically closed. Then each birational transformation of the coordinates p_1, p_2 homogeneous of degree 1, which doesn't depend on the coordinates q_1, q_2 can be prolonged to the affine transformation of the coordinates q_1, q_2. This transformation of* $\mathrm{Symp}(J^1\mathbb{k})$ *projects on a unique affine p-map.*

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INVERSE COEFFICIENT PROBLEMS FOR NONLINEAR CONVECTION–DIFFUSION REACTION EQUATION

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Inverse problems' research for linear and nonlinear heat-and-mass transfer models has been an urgent issue during a long period of time. One of the main problems is an identification problem for unknown densities of boundary and distributed sources or of model's differential equations' coefficients or boundary conditions with the help of additional information about system's conditions, which is described by the model.

In a bounded domain $\Omega \subset \mathbb{R}^3$ with the boundary Γ the following boundary value problem is considered

$$-\mathrm{div}(\lambda(\mathbf{x})\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi + k(\varphi,\mathbf{x})\varphi = f \text{ in } \Omega, \quad \varphi = \psi \text{ on } \Gamma. \tag{1}$$

Here function φ means polluting substance's concentration, $\mathbf{u}$ is a given vector of velocity, f is a volume density of external sources of substance, $\lambda(\mathbf{x})$ – a diffusion coefficient, function $k(\varphi,\mathbf{x})$ is a reaction coefficient, ψ is a given boundary function.

About solvability of boundary value and extremum problems for nonlinear convection–diffusion–reaction equation in case when reaction coefficient depends on either substance's concentration or on spatial variables and functions belonging to a rather wide class. see [2, 3]. Then an identification problem is formulated for the reaction coefficients of the form of $\tilde{k} = \beta(\mathbf{x})\tilde{k}(\varphi)$, which consists in functions $\beta(\mathbf{x})$ and $\lambda(\mathbf{x})$ recovering, using the measured substance's concentration in a subdomain $Q \subset \Omega$. At that, the conditions on functions $\beta(\mathbf{x}), k(\varphi)$ and $\lambda(\mathbf{x})$ are also rather general.

This work was supported by the Russian Foundation for Basic Research (project no. 16-11-00365-a).

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THE NUMBER OF SOLVENTS OF SECOND-ORDER UNILATERAL MATRIX POLYNOMIALS OVER PRIME FINITE FIELDS

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Unilateral matrix polynomial (UMP) of n-th order and d-th degree over field $\mathbb{K}$ is the expression of the form

$$\mathcal{F}(X) = X^d + \mathbf{F}_{d-1} \cdot X^{d-1} + \cdots + \mathbf{F}_2 \cdot X^2 + \mathbf{F}_1 \cdot X + \mathbf{F}_0, \tag{1}$$

where $\mathbf{F}_i \in \mathsf{M}_n(\mathbb{K}), i = 0, ..., d-1$ are coefficients and $X \in \mathsf{M}_n(\mathbb{K})$ is variable.

The *solvent* (or *zero*) of (1) over $\mathbb{K}$ is a matrix $\mathbf{S} \in \mathsf{M}_n(\mathbb{K})$, such that $\mathcal{F}(\mathbf{S}) = \mathbf{0}$, where $\mathbf{0}$ is n-th order zero matrix.

The problem of finding and counting solvents of (1) has been extensively studied in for the case, when $\mathbb{K}$ is $\mathbb{C}$, the set of complex numbers [1, 2, 3, 4]. Also there are some works for the case, when $\mathbb{K}$ is finite field, but for very specific forms of (1) (see, for example, [5, 6]).

This work fills the gap in counting solvents of arbitrary second-order UMP over prime finite field $\mathbb{F}_p$. In particular there is estimations of upper/lower bound of all solvents for given UMP, along with specific solvents such as diagonalizable or having non-empty spectrum over $\mathbb{F}_p$.

The work is supported by RFBR grant 16-37-00125 mol_a.

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ON THE RANK OF ODD HYPER-QUASI-POLYNOMIALS

V. A. Bykovskii (IAM FEB RAS, Khabarovsk)

For a fixed non-zero entire function $g : \mathbb{C} \to \mathbb{C}$, we define linear complex space $\mathcal{F}(g)$. It consist of all entire functions f such that the following decomposition is valid

$$f(z+w)g(z-w) = \varphi_1(z)\psi_1(w) + \cdots + \varphi_n(z)\psi_n(w)$$

for some $\varphi_1, \psi_1, \ldots, \varphi_n, \psi_n : \mathbb{C} \to \mathbb{C}$. The rank of f with respect to g is a minimal n such that the decomposition is possible. We prove that if g is an odd function, then the rank of any function from $\mathcal{F}(g)$ is an even integer.

This research is supported by the Russian Science Foundation (project №14-11-00335).

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EXPONENTIAL STABILITY OF STATIONARY SOLUTIONS OF RADIATION HEAT TRANSFER EQUATIONS

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The conventional non-stationary normalized P_1 approximation of the complex heat transfer model describing radiative and conductive contributions is considered in a bounded domain Ω with boundary Γ. The model has the following form (cf. [1]):

$$\partial\theta/\partial t - a\Delta\theta + b\kappa_a\theta^4 = b\kappa_a\varphi, \quad -\alpha\Delta\varphi + \kappa_a\varphi = \kappa_a\theta^4,$$

$$a\partial_n\theta + \beta(\theta - \theta_b)|_\Gamma = 0, \quad \alpha\partial_n\varphi + \gamma(\varphi - \theta_b^4)|_\Gamma = 0, \quad \theta|_{t=0} = \theta_0.$$

Here, θ is the normalized temperature, φ the normalized radiation intensity averaged over all directions. Parameters $a > 0$, $\alpha > 0$, $b > 0$, and the absorption coefficient κ_a describe the radiation-thermal properties of the medium.

Analysis of complex heat transfer in scattering media with reflecting boundaries is important for applications. A lot of work is connected with the numerical simulation of complex heat transfer processes in continuous media. At the same time, few papers are devoted to the theoretical analysis of the corresponding boundary value problems, that allows to assess the adequacy of the models of radiative heat transfer.

The main result of this work is obtaining new a priori estimates of temperature and radiation intensity, that enables to prove the exponential stability of stationary solutions of radiative heat transfer equations.

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ON HYPERBOLIC HURWITZ ZETA FUNCTION

N. M. Dobrovolsky, N. N. Dobrovolsky, V. N. Soboleva, D. K. Sobolev, L. P. Dobrovol'skaya, O. E. Bocharova

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The paper deals with a new object of study — hyperbolic Hurwitz zeta function, which is given in the right α-semiplane $\alpha = \sigma + it$, $\sigma > 1$ by the equality

$$\zeta_H(\alpha; d, b) = \sum_{m\in\mathbb{Z}} \left(\overline{dm+b}\right)^{-\alpha},$$

where $d \neq 0$ and b — any real number.

Hyperbolic Hurwitz zeta function $\zeta_H(\alpha; d, b)$, when $\left\|\frac{b}{d}\right\| > 0$ coincides with the hyperbolic zeta function of shifted one-dimensional lattice $\zeta_H(\Lambda(d,b)|\alpha)$. The importance of this class of one-dimensional lattices is due to the fact that each Cartesian lattice is represented as a union of a finite number of Cartesian products of one-dimensional shifted lattices of the form $\Lambda(d,b) = d\mathbb{Z} + b$.

Cartesian products of one-dimensional shifted lattices are in substance shifted diagonal lattices, for which in this paper the simplest form of a functional equation for the hyperbolic zeta function of such lattices is given.

The connection of the hyperbolic Hurwitz zeta function with the Hurwitz zeta function $\zeta^*(\alpha; b)$ periodized by parameter b and with the ordinary Hurwitz zeta function $\zeta(\alpha; b)$ is studied.

New integral representations for these zeta functions and an analytic continuation to the left of the line $\alpha = 1 + it$ are obtained.

All considered hyperbolic zeta functions of lattices form an important class of Dirichlet series directly related to the development of the number-theoretical method in approximate analysis. For the study of such series the use of Abel's theorem is efficient, which gives an integral representation through improper integrals. Integration by parts of these improper integrals leads to improper integrals with Bernoulli polynomials, which are also studied in this paper.

THE COMBINATORY K-ARY GCD

D. A. Dolgov (Kazan (Volga region) Federal University, Kazan)

The greatest common divisor (GCD) of two integers is a basic arithmetic operation used in many mathematical software systems. Sorenson introduced the k-ary gcd algorithm (see [1]), a generalization of the binary algorithm.

Given positive integers $u > v$ relatively prime to k, integers a and b can be found that satisfy $au + bv = 0 \pmod{k}$ with $|a|,\ |b| < \sqrt{k}$. If we set $a = 1, b = -1, k = 2$ we have the binary alrorithm as a special case.

Weber improved the k-ary gcd algorithm [3]. He described an algorithm to find a and b that replaces precomputed tables. He presented the *dmod* operation in the main loop to adjust the size of input numbers.

Our goal is to reduce the number of operations of the k-ary gcd. We use improved Weber's version. We offer to take t last bits of input numbers to choose the coefficients of algorithm. We constuct linear transformation bitwise $u_t,\ v_t$ [2]:

$$(\alpha_t * u_t \pm \beta_t * v_t)/2^s,$$

where s is a max power of 2 that $(\alpha_t * u_t \pm \beta_t * v_t) = 0 \pmod{2^s}$, u_t, v_t are the last t bits of input numbers u, v. To find α_t, β_t we compute C_i sequences:

$$C_i = (max(A_i, B_i) \pm min(A_i, B_i))/2^{s_i},\ A_i = min(A_{i-1}, B_{i-1}),\ B_i = C_{i-1}$$

where $A_1 = \max(u_k, v_k)$, $B_1 = \min(u_k, v_k), i > 1$. At each step we find s_i, which is max power of 2, that C_i is integer. We construct sequence of $A_i,\ B_i,\ C_i$ while $C_i > 0$. Exit conditions is $C_i = 0$. C_i sequence may be calculated by adding or subtracting elements. We choose everywhere plus or everywhere minus operation. At the end $a = \alpha_t, b = \beta_t$.

To reduce the number of operations we analyze the value of t parameter. Choosing as t the maximum possible value, we do not always get the minimum number of iterations.

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GCD AS A SOLUTION OF SYSTEM OF LINEAR EQUATIONS IN GF(2)

D. A. Dolgov (Kazan (Volga region) Federal University, Kazan)

Computation of the greatest common divisor (GCD) is used in many cryptographic algorithms. It can be factorization algorithms, such as the $p-1$ Pollard algorithm, the Lenstra algorithm.

We compute gcd of 2 natural numbers. Any number can be represented as a result of computation the value of a unitary polynomial. We shall assume that the digits of a number are the coefficients of this polynomial. If polynomials aren‘t coprime, so the values of this polynomials aren‘t coprime too for fixed input parameter. Let we have 2 polynomials $f(x), g(x)$. We use Bezout identity with some coefficient polynomials $u(x), v(x)$:

$$u(x) * f(x) + v(x) * g(x) = d(x) = gcd.$$

It can be represented in a matrix representation. The quadratic Sylvester matrix A is obtained. We get a system of linear equations (slu). To solve the slu we compute rank (rk) of the Sylvester matrix. After that we choose $rk-1$ rows. The number of rows does not match the number of variables. The last $max(deg(f), deg(g)) + deg(g) - rk + 2$ variables are chosen randomly. Then we solve the slu $A_{rk-1} * X = 0$, where A_{rk-1} is a matrix consists of the first $rk-1$

of the matrix A. Result is not a clear polynomial gcd. Result has some spurious factors. So, we must combine this method with exact gcd algorithms like Stein gcd or Euclid gcd. At we end we compute numeric gcd as a result of unitary polynomial for fixed x.

Computing over $GF(2)$ can speed up the gcd computation. For example, to calculate the rank of a matrix, the solution of the slu. This algorithm can be well parallelized because there are parallel algorithms for computing the rank of the matrix and the solution of the system.

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CONNECTED LEMNISCATES AND DISTORTION THEOREMS FOR POLYNOMIALS AND RATIONAL FUNCTIONS

V. N. Dubinin (Institute of Applied Mathematics Far Eastern Branch of the Russian Academy of Science, Vladivostok)

In the present talk we discuss the impact of the connectivity of some lemniscates of a rational function f on the distortion of the mapping effected by f. We consider the distortion theorems for polynomials and rational functions [1]-[2], an inequality for the moduli of derivative of a complex polynomial at its zeros [3] and an inequality for the logarithmic energy of zeros and poles of a rational function [4]. All estimates obtained are sharp. The proofs of the theorems go by an application of a certain modification of the symmetrization method [5], for which the result of the symmetrization lies on the Riemann surface of the inverse function to a Chebyshev polynomial of the first kind.

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NON-VANISHING OF AUTOMORPHIC L-FUNCTIONS OF PRIME POWER LEVEL

D. A. Frolenkov (Steklov Mathematical Institute of RAS, Moscow, Russia)

Iwaniec and Sarnak showed that at the minimum 25% of L-values associated to holomorphic newforms of fixed even integral weight and level $N \to \infty$ do not vanish at the critical point when N is square-free and $\phi(N) \sim N$. We extend the given result to the case of prime power level $N = p^{\nu}$, $\nu \geqslant 2$. The proof is based on asymptotic evaluation of twisted moments and the technique of mollification.

This is a joint work with Olga Balkanova.

ON NUMBER-THEORETIC MULTIDIMENSIONAL MATRICES AND DETERMINANTS

A. S. Gasparyan (Program System Institute of RAS, Pereslavl-Zalesskii, Russia)

H.J.S. Smith (1875) considered the GCD-matrix $[(i,j)]$, $i,j = 1,\ldots,n$, where (a,b) denotes the greatest common divisor of a,b. Smith showed that $det[(i,j)] = \phi(1)\cdots\phi(n)$, ϕ being Euler's totient. Since that time Smith's determinant and related identity were generalized in several directions , particularly for factor-closed sets, for matrices of type $[f((x_i,x_j))]$, and for so called meet matrices on posets. But most interesting extension was a generalization of Smith's determinantal identity by P.Haukkanen (1992) to multidimensional matrices $[(x_{i_1},\ldots,x_{i_p})]$, $i_1,\ldots,i_p = 1,\ldots,n$, when $S = \{x_1,\ldots,x_n\}$ is a factor-closed set.

In this talk we present another generalizations of results relating GCD-type matrices and determinants to multidimensional ones. Here we give an example.

Let $X^{(r)} = \{x_1^{(r)}, \dots, x_n^{(r)}\}$, $r = 1, \dots, p, x_i^{(r)} \in N$, let $S = \{y_1, \dots, y_m\}$ be minimal factor-closed set containing $X^{(1)}, \dots, X^{(p)}$.

Theorem. *There holds the identity*

$$\left|(x_{i_1}^{(1)}, \dots, x_{i_p}^{(p)})\right|^{(\sigma_1, \dots, \sigma_p)} = \sum_{1 \leqslant \alpha_1 < \dots < \alpha_n \leqslant m} \phi(y_{\alpha_1}) \cdots \phi(y_{\alpha_n}) \prod_{r=1}^{p} |C^{(r)}[1, \dots, n|\alpha_1, \dots, \alpha_n]|^{(\sigma_r, \sigma_r)},$$

where $\vec{\sigma} = (\sigma_1, \dots, \sigma_p) \in \{0,1\}^p, |\vec{\sigma}| = 2d, 1 \leqslant d \leqslant [p/2]$, $\vec{\sigma}$ *is the signature of p-dimensional determinant in the left hand of identity, and* $C^{(r)} = [c_{ij}^{(r)}]$ *is a divisibility matrix defined as* $c_{ij}^{(r)} = 1$ *if* $y_j | x_i^{(r)}$, *and* $= 0$ *otherwise.*

Corollary. *The following inequality is true.*

$$\left|(x_{i_1}^{(1)}, x_{j_1}^{(1)} \dots, x_{i_k}^{(k)}, x_{j_k}^{(k)})\right|^{(\sigma_1, \sigma_1 \dots, \sigma_k, \sigma_k)} \geqslant 0.$$

[1] H. J. S. Smith On the value of a certain arithmetical determinant // Proc. London Math. Soc., **7** (1875–76), 208–212.

DIOPHANTINE EXPONENTS AND PRODUCTS OF LINEAR FORMS

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Let $\mathcal{L}_d$ denote the space of unimodular lattices of full rank in $\mathbb{R}^d$. Let $\Lambda \in \mathcal{L}_d$. For each $\boldsymbol{v} = (v_1, \dots, v_d)$ set

$$\Pi(\boldsymbol{v}) = |v_1 \cdot \ldots \cdot v_d|^{1/d}.$$

If Λ is algebraic, then $\Pi(\boldsymbol{v})$ is bounded away from zero at nonzero lattice points. The same holds if Λ is the image of an algebraic lattice under the action of a non-degenerate diagonal matrix. It is an open question whether other such lattices exist for $d \geqslant 3$. A negative answer to this question is known to imply the Littlewood conjecture.

We turn our attention to a more general situation, when $\Pi(\boldsymbol{v})$ can attain however small values. In this case it is reasonable to talk about the rate of tending $\Pi(\boldsymbol{v})$ to zero over a sequence of lattice points, which leads to the notion of a Diophantine exponent of Λ defined as

$$\omega(\Lambda) = \sup \left\{ \gamma \in \mathbb{R} \;\middle|\; \Pi(\boldsymbol{v}) < |\boldsymbol{v}|^{-\gamma} \text{ admits } \infty \text{ solutions in } \boldsymbol{v} \in \Lambda \right\}.$$

In our talk we discuss all that is currently known about Diophantine exponents of lattices and pay special attention to the spectrum

$$\Omega_d = \left\{ \omega(\Lambda) \;\middle|\; \Lambda \in \mathcal{L}_d \right\}.$$

Particularly, we show how to prove that for each d there is a positive λ such that the ray $[\lambda, +\infty]$ is contained in Ω_d. To this end we present a new existence theorem concerning linear forms of a given Diophantine type.

CANONICAL DIAGRAMS FOR LATTICES IN DIMENSION THREE AND PACKINGS OF A PLANE

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The connection between canonical Voronoi diagrams for three dimensional maximal lattices repeating by multiplication and special periodic packing of a plane is given. For example, it is shown: the canonical Voronoi diagram is a hexagon tessellation when the unit of the lattice has only six adjacent minimums. The algorithm to construct the periodic packing of canonical Voronoi diagram is given.

SPATIAL DISTRIBUTION OF CONJUGATE ALGEBRAIC NUMBERS

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We consider the distribution of points whose coordinates are conjugate algebraic numbers of a fixed degree.

Let $q(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial of degree n. For a fixed vector of positive weights $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ and a fixed real $p \in [1, \infty]$, define the height of q as

$$h(q) = h_{p,\mathbf{w}}(q) := \left(\sum_{k=0}^{n} |w_k a_k|^p \right)^{1/p}.$$

Note if $w_i = 1$ and $p = \infty$, then $h(q) = \max_{0 \leqslant k \leqslant n} |a_k|$ is the well-known naïve height. The degree $\deg \alpha$ and height $h(\alpha)$ of an algebraic number α are defined as the degree and height of the minimal polynomial of α over $\mathbb{Z}$.

Let fixed integers $k, l \geqslant 0$ satisfy $0 < k + 2l \leqslant n$. Denote $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For a set $B \subset \mathbb{R}^k \times \mathbb{C}_+^l$ and the height function h, denote by $\Phi_{k,l}(Q, B)$ the number of ordered mixed (k,l)-tuples $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l) \in B$, where real α_i and complex β_j all are distinct algebraic and share the same minimal polynomial over $\mathbb{Z}$ of degree n and height at most Q (that is, α_i and β_j all are conjugates).

Theorem ([1]). *There exists a function $\rho_{k,l} : \mathbb{R}^k \times \mathbb{C}_+^l \to \mathbb{R}_+$ such that for any fixed Jordan measurable $B \subset \mathbb{R}^k \times \mathbb{C}_+^l$, we have*

$$\lim_{Q \to \infty} \left| \frac{\Phi_{k,l}(Q, B)}{Q^{n+1}} - \gamma_n \int_B \rho_{k,l}(\mathbf{v}) \, \mathrm{d}\mathbf{v} \right| = 0, \tag{1}$$

where γ_n is an explicit constant depending only on n and the function h; here $\mathrm{d}\mathbf{v}$ is a volume element in $\mathbb{R}^k \times \mathbb{C}^l \cong \mathbb{R}^{k+2l}$.

Note that $\rho_{k,l}$ depends on the height function h.

If the boundary ∂B is smooth enough, we are able to estimate the rate of convergence in (1). For example, if ∂B is the union of m algebraic surfaces of degree at most d, then for $n \geqslant 3$ the rate of convergence in (1) can be estimated as $C_{n,m,d}\, Q^{-1}$, where $C_{n,m,d}$ depends only on n, m, d and on the height function h.

It turns out that the function $\rho_{k,l}$ coincides with the correlation function of the roots of some specific random polynomial. Moreover, there exist explicit symmetric functions $\psi_m : \mathbb{C}^m \to \mathbb{R}_+$ (depending on n) such that all the functions $\rho_{k,l}$ satisfy the equality

$$\rho_{k,l}(x_1, \ldots, x_k, z_1, \ldots, z_l) = 2^l \psi_{k+2l}(x_1, \ldots, x_k, z_1, \ldots, z_l, \overline{z}_1, \ldots, \overline{z}_l).$$

Every ψ_m can be represented as the product of $\prod_{1 \leqslant i < j \leqslant m} |z_i - z_j|$ and a continuous function.

[1] F. Götze, D.V. Koleda, D.N. Zaporozhets. Joint distribution of conjugate algebraic numbers: a random polynomial approach // arXiv:1703.02289, 2017.

ON PELL IDENTITIES WITH MULTINOMIAL COEFFICIENTS

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The *Pell numbers* are an integer sequence defined recursively by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geqslant 2$ (see [1] and the references given there).

Proposition. *Let $n \geqslant 1$, except when otherwise. The following formulas hold*:

$$\sum_{(t_1,\ldots,t_n)} (-1)^T p_n(t) P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n} = -F_n,$$

$$\sum_{(t_1,\ldots,t_n)} (-1)^T p_n(t) P_1^{t_1} P_3^{t_2} \cdots P_{2n-1}^{t_n} = -4 \cdot 5^{n-2}, \quad n \geqslant 2,$$

$$\sum_{(t_1,\dots,t_n)} (-1)^T p_n(t) P_2^{t_1} P_4^{t_2} \cdots P_{2n}^{t_n} = \frac{(2-\sqrt{3})^n - (2+\sqrt{3})^n}{\sqrt{3}},$$

$$\sum_{(t_1,\dots,t_n)} (-1)^T p_n(t) P_3^{t_1} P_5^{t_2} \cdots P_{2n+1}^{t_n} = -4, \quad n \geqslant 2,$$

$$\sum_{(t_1,\dots,t_n)} (-1)^T p_n(t) P_2^{t_1} P_3^{t_2} \cdots P_{n+1}^{t_n} = 0, \quad n \geqslant 3,$$

$$\sum_{(t_1,\dots,t_n)} (-1)^T p_n(t) P_0^{t_1} P_1^{t_2} \cdots P_{n-1}^{t_n} = -2^{n-2}, \quad n \geqslant 2,$$

$$\sum_{(t_1,\dots,t_n)} (-1)^T p_n(t) P_3^{t_1} P_4^{t_2} \cdots P_{n+2}^{t_n} = (-1)^n F_{2n+3},$$

$$\sum_{(t_1,\dots,t_n)} p_n(t) P_0^{t_1} P_1^{t_2} \cdots P_{n-1}^{t_n} = \frac{(1+\sqrt{3})^{n-1} - (1-\sqrt{3})^{n-1}}{2\sqrt{3}},$$

$$\sum_{(t_1,\dots,t_n)} p_n(t) P_0^{t_1} P_2^{t_2} \cdots P_{2n-2}^{t_n} = \frac{(3+\sqrt{10})^{n-1} - (3-\sqrt{10})^{n-1}}{\sqrt{10}},$$

where the summation is over integers $t_i \geqslant 0$ *satisfying* $t_1 + 2t_2 + \cdots + nt_n = n$, $T = t_1 + \cdots + t_n$, F_n *is the* n*-th Fibonacci number, and* $p_n(t) = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ *is the multinomial coefficient.*

[1] T. Koshy. Pell and Pell-Lucas Numbers with Applications. Springer, 2014.

DISTRIBUTION OF VALUES JORDAN'S FUNCTION IN RESIDUE CLASSES

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The pair of integers a, b is called primitive modulo integer n if the greatest common divisor $(a, b, n) = 1$. Let's denote by $J(n)$ the number of incongruent primitive pairs of integers modulo n. This function is called Jordan's function. The properties of $J(n)$ were studied in [1].

Let's denote by $S(x, r, f)$ for $x > 0$, integers r, N, $(r, N) = 1$, and integral-value arithmetical function $f(n)$ the number of integers $n > 0$ for which $f(n) \equiv r$

(mod N). According to W. Narkiewicz [2] a function $f(n)$ is called weakly uniformly distributed modulo N if and only if for every pair of integer a, b coprime with N $\lim_{x\to\infty} S(x,a,f)S^{-1}(x,b,f) = 1$ provided that the set $\{n|(f(n),N)\}$ is infinite.

This information is provided in our report to the next results.

Let χ — Dirichlet character modulo N; χ_0 — principal Dirichlet character; $G(N)$ — the group of residues coprime with N; $\varphi(n)$ — Euler function.

Theorem 1. *The functon $J(n)$ weakly uniformly distributed modulo N if and only if $(N,6) = 1$.*

Theorem 2. *If $(N,6) = 1$, then for any integer $n > 0$ and for any $r \in G(N)$ for $x \to \infty$*

$$S(x,r,J) = \frac{x}{\varphi(N)\ln x}\left(\sum_{k=0}^{n-1}\frac{a_k}{(\ln x)^{k-\lambda}} + \sum_{\chi\neq\chi_0}\sum_{k=0}^{n-1}\frac{b_k(\chi)}{(\ln x)^{k-\xi(\chi)}}\right) + $$
$$+O\left(\frac{\Gamma(n+1-\xi_0)x}{(\ln x)^{n+1-\xi_0}}\right),$$

where $\lambda \in (0,1)$, $\xi(\chi) \in (-\lambda,\lambda)$, $\xi_0 = \max_{\chi}\xi(\chi)$, a_k, $b_k(\chi)$ — the complex numbers.

[1] J. Shulte. Uber die Jordansche Verallgemeinerung der Eulerschen Funktion // Resultate der Mathematik, **36**:3–4 (1999).

[2] W. Narkievicz. On distribution of values of multiplicative functions in residue classes // Acta Arithm., **12**:3 (1967).

HYPERQUASIPOLYNOMIALS FOR THETA FUNCTION

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We consider the functional equation

$$f(x+y)g(x-y) = \sum_{k=1}^{r}\alpha_k(x)\beta_k(y). \tag{1}$$

Here f, g, α_k, β_k are unknown functions $\mathbb{C} \to \mathbb{C}$. A general solution of the equation (1) is known only for $r = 1, 2$ (see [1]), and $r = 3$ (see [2, 3]).

Let g be a fixed nonzero entire function and $\mathcal{F}(g)$ be the set of functions f such that there exist natural r, functions α_k, β_k for which the expansion (1) holds. If $g \equiv 1$ then the set $\mathcal{F}(g)$ consists of quasipolynomials, that is

$$f(x) = \sum_{k=1}^{s} P_k(x) e^{\lambda_k x},$$

where $\lambda_k \in \mathbb{C}$ and P_k is a polynomial.

Definition ([4]). Any function from $\mathcal{F}(g)$ is called a hyperquasipolynomial with respect to the function f.

Let $V(g)$ be the set of all functions f of the form

$$f(z) = \sum_{j=1}^{n} \sum_{k=0}^{m_j} Q_{jk}(z) g^{(k)}(z + z_{jk}),$$

where $n > 0$, $m_j \geqslant 0$ are integer, Q_{jk} is a quasipolynomial, $z_{jk} \in \mathbb{C}$, and $g^{(k)}$ denotes k-th derivative of g.

We prove the following result.

Theorem. Let θ be a Jacobi theta function. Then

$$\forall g \in V(\theta) \backslash \{0\} \qquad \mathcal{F}(g) = V(\theta).$$

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[3] A. A. Illarionov. Functional equations concerning with addition theorems for elliptic functions // Proceedings of the Steklov Institute of Mathematics, **299** (2017).

[4] V.A. Bykovskii. Hyperquasipolynomials and their applications // Functional Analysis and Its Applications, **50**:3 (2016), 193–203.

AN ANALOGUE OF KHINCHINE'S THEOREM FOR THE DIVERGENCE CASE IN THE SPACE $\mathbb{Q}_p^2$

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In 1924 Khinchine proved the following theorem, which gave rise to a number of problems in the metric theory of Diophantine approximation. Let $\Psi(x)$ be a monotonic decreasing function defined on $\mathbb{R}_+$, and let I be an interval in $\mathbb{R}$. Let $L_1(\Psi)$ be a set of all real numbers $x \in I$ satisfying the inequality

$$|x - p/q| < \Psi(q)/q$$

for infinitely many numbers $p, q \in \mathbb{Z}$ c $q \neq 0$.

Theorem 1 (Khinchine, [1]).

$$\mu\left(L_1(\Psi)\right) = \begin{cases} 0 & \sum_{r=1}^{\infty} \Psi(r) < \infty \\ \mu(I) & \sum_{r=1}^{\infty} \Psi(r) = \infty \end{cases}.$$

For polynomials, a natural extension of Khinchine's theorem was formulated as follows. Let $L_n(\Psi)$ denote the set of points $x \in \mathbb{R}$ such that the inequality

$$|P(x)| < H^{-n+1}\Psi(H)$$

has infinitely many solutions in polynomials $P \in \mathbb{Z}[x]$, where H is the height of polynomial P and $\deg P = n$. An analogue of Khinichine's theorem for $L_n(\Psi)$ has been proved in the articles [2, 3] for convergence and divergence respectively. Similar results have also been obtained for complex and p-adic numbers [4, 5]. Several extensions of Khinchine's theorem have also been considered in spaces of higher dimensions.

Let Ψ be a monotone decreasing positive function and let the series

$$\sum_{r=1}^{\infty} \Psi(r)$$

be divergent.

Let $T = D_1 \times D_2$, where $D_1, D_2 \subset \mathbb{Q}_p$ are cylinders in $\mathbb{Q}_p$, and let $\mathcal{L}_{\mathrm{v},\lambda}$ be the set of points $(\omega_1, \omega_2) \in T$ such that the system of inequalities

$$\begin{cases} |P(\omega_1)|_p < H^{-v_1}\, \Psi^{\lambda_1}(H) \\ |P(\omega_2)|_p < H^{-v_2}\, \Psi^{\lambda_2}(H) \end{cases} \tag{1}$$

has infinitely many solutions in polynomials $P \in \mathbb{Z}[x]$, where

$$v_1 + v_2 = n - 2 \text{ и } \lambda_1 + \lambda_2 = 1. \tag{2}$$

The following theorem has been proved.

Theorem 2. *Under the conditions* (1) *and* (2), *if* $n \geqslant 2$ *and* $\sum_{r=1}^{\infty} \Psi(r) = \infty$, *then*

$$\mu \mathcal{L}_{\mathrm{v},\lambda} = \mu(T).$$

The proof of Theorem 2 is based on a regular system of vectors with algebraic coordinates in $\mathbb{Q}_p^2$. A similar system for a different multi-dimensional case has been constructed in [6], leading to analogues of Khinchine's theorem in the space $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ [6, 7].

[1] A. Khintchine. Einige Sätze über Kettenbrüche mit Anwen auf die Theorie der Diophantischen Approximationen // Math. Ann., **92**:1-2 (1924), 115–125.

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COVERING THEOREM FOR MULTIVALENT FUNCTIONS

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Let $M_p(\omega)$, $p \geqslant 2$, be the class of all holomorphic p-valent functions in the unit disk $U = \{z : |z| < 1\}$ with normalization $f(0) = 0$, $f(\omega) = \omega$, $0 < \omega < 1$. [1, Ch. 3.4]. Let $\mathcal{R}(f)$ denote the Riemann surface obtained as image of the unit disk $U = \{z : |z| < 1\}$ under the mapping f of class $M_p(\omega)$. For every function from the class $M_p(\omega)$, we obtain the maximal value $\rho(p, \omega)$, for which the Riemann surface $\mathcal{R}(f)$ contains an open k-valent disk, $k \leqslant p$, branching over the disk $|w| < \rho(p, \omega)$.

$$\rho(p, \omega) := \frac{\omega}{T_p\left[\frac{4\omega+(1+\omega)^2\cos(\pi/(2p))}{(1-\omega)^2}\right]},$$

where $T_p(z) = 2^{p-1}z^p + \ldots$ is the Chebyshev polynomial of the first type.

[1] A. Vasil'ev. Moduli of Families of Curves for Conformal and Quasiconformal Mappings, Lecture Notes in Math., **1788**, Springer-Verlag, Berlin, New York, 2002.

SMALL VALUES OF IRREDUCIBLE DIVISORS OF INTEGER POLYNOMIALS

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We present an improvement to the A.O. Gelfond's lemma on the order of zero approximation by irreducible divisors of integer polynomials [1, lemma VI, p. 183]. The lemma says that if a polynomial $P(x) \in \mathbb{Z}[x]$ of degree not exceeding n and of height not exceeding Q satisfies inequality $|P(x)| < Q^{-w}$, $w > 6n$, for some transcendental point $x \in \mathbb{R}$, then there exists a divisor $d(x) \in \mathbb{Z}[x]$ of $P(x)$, which can be written as a degree of some polynomial irreducible over the field of rational numbers, satisfying $|d(x)| < Q^{-w+6n}$.

Gelfond's lemma and similar results have important applications to many problems of the metric theory of Diophantine approximation. One of such applications is the result of V.I. Bernik [2] on the upper bound for the Hausdorff dimension of the set of real numbers with specified order of zero approximation by the values of integer polynomials. This result along with the result of A. Baker and W.M. Schmidt [3] on the lower bound for the Hausdorff dimension of the set mentioned above gives the exact formula.

In order to prove the upper bound V.I. Bernik improved and extended Gelfond's lemma by using a weaker condition $w > 3n$ and obtaining a better estimate $|d(x)| < Q^{-w+n}$ as well as considering the values of polynomials on an interval. However, the condition on w is still restrictive and limits the range of problems this result could be applied to.

We improve the existing results by obtaining the estimate $|d(x)| < Q^{-w+n-1}$ on some interval for any w.

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UNIVERSALITY OF ZETA-FUNCTIONS OF CUSP FORMS

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Suppose that $F(z)$ is a normalized Hecke eigen cusp form of weight κ for the full modular group, and let $c(m)$, $m \in \mathbb{N}$, be its Fourier coefficients. The zeta function $\zeta(s, F)$, $s = \sigma + it$, of the form $F(z)$, for $\sigma > \frac{\kappa+1}{2}$, is defined by the series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and can be analytically continued to an entire function.

In the report, we consider the approximation of wide classes of analytic functions by shifts $\zeta(s + i\tau, F)$, $\tau \in \mathbb{R}$. The main attention is devoted to the so-called discrete universality, when τ in $\zeta(s+i\tau, F)$ takes values from a certain discrete set. For example, the following theorem is true [1].

Theorem 1. *Suppose that K is a compact subset of the strip $\left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$ with connected complement, and $f(s)$ is a continuous non-vanishing function on K that is analytic in the interior of K. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh, F) - f(s)| < \varepsilon \right\} > 0.$$

A joint version of Theorem 1 is also considered.

[1] A. Laurinčikas, K. Matsumoto, J. Steuding. Discrete universality of L-functions of new forms. II // Lith. Math. J., **56** (2016), 207–218.

P-HARMONIC MAPPINGS OF SPATIAL DOMAINS AND THE INNER P-HARMONIC RADIUS

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The class of mappings is considered for which level surfaces $\{x \in E^n : u_G(x, x_0) = \nu\}$ p-harmonic Green's function $u_G(x, x_0)$ of the domain $G \subset E^n$ ([1]) are displayed in the level surface $\{y \in E^n : u_{G^*}(y, y_0) = \nu\}$ p-harmonic Green's function $u_{G^*}(y, y_0)$ of the domain $G^* \subset E^n$, and the trajectory of the gradient field *grad* $u_G(x, x_0)$, which enters the pole x_0, corresponds to the trajectory of the field *grad* $u_{G^*}(y, y_0)$, which enters the pole y_0 $(1 < p < \infty)$. The construction of such mappings is carried out by analogy with the harmonic mappings M.A. Lavrentyev, considered in the monograph A.I. Janushauskas [2] $(p = n = 3)$. The properties of such mappings are established and, in particular, the following theorem is proved.

Let the domains $G \subset E^n$ and $G^* \subset E^n$ be such that their p-harmonic Green's functions have no critical points and have the property that for any ray l, issuing from the point $x_0 \in G$ (respectively $y_0 \in G^*$), there is the unique trajectory of the field *grad* $u_G(x, x_0)$ (respectively, the unique trajectory of the field *grad* $u_{G^*}(y, y_0)$), entering x_0 (respectively y_0) with the tangent l. Let $f : G \to G^*$ - p-harmonic mapping of the domain G on the domain G^*, $f(x_0) = y_0$, and Jacobian of the mapping f for $p \leqslant n$ in a neighborhood of a pole x_0 can be represented in the form $I(f) = 1 + O(|x - x_0|^{\gamma+\alpha})$, if $x \to x_0$, $\gamma = \frac{n-p}{p-1}$, $\alpha > 0$. Then for every p $(1 < p < \infty)$ the inner p-harmonic radius ([3]) of the domain G^* in y_0 is equal to the inner p-harmonic radius of the domain G in x_0.

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[2] A.I. Yanushauskas, Three-dimensional analogs of conformal mappings // Novosibirsk, Nauka, 1982, 173 (in Russian).

[3] B. Levitskii, Reduced p-modulus and the inner p-harmonic radius// Dokl. Akad. Nauk SSSR, **316**:4 (1991), 812–815 (in Russian); translation in: Soviet Math. Dokl., **43**:1 (1991), 189–192.

NUMERICAL ANALYSIS OF 2D ELECTROMAGNETIC CLOAKING PROBLEM

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In recent years, much attention has been given to study of inverse problems of wave scattering theory arising while developing devices for cloaking material objects. Beginning with the paper [1], the large number of publications was devoted to developing different methods of solving cloaking problems (see [2] and references therein). We also mention papers [3]–[5] which are devoted to theoretical analysis of cloaking problems using the optimization approach.

In this paper we consider cloaking problem for the electromagnetic scattering model described by the 2D Helmholtz equation and develop effective numerical algorithm for solving the cloaking problem under consideration. This algorithm is used to numerically modeling the cloaking properties of a multilayer cylindrical shell consisting of concentric layers filled with a homogeneous anisotropic medium. On the basis of the methods "Singular Value Decomposition" and "Particle Swarm Optimization", a numerical algorithm was developed whose purpose is to find a set of constant parameters of the medium in each layer of the cloaking shell. These parameters are selected by solving the minimization problem. The properties of the algorithm are studied and some results of numerical experiments are discussed.

This work was supported by the Russian Science Foundation (project no. 14-11-00079).

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APPROXIMATION BY DISCRETE SHIFTS OF RIEMANN ZETA-FUNCTION

R. Macaitienė (Siauliai University, Siauliai State College, Šiauliai, Lithuania)

Universality (in the Voronin sense) of zeta and L-functions is one of the most interesting phenomenons of analytic number theory. It means that analytic on compact sets functions can by approximated by shifts of some zeta-functions and, in general, of Dirichlet series. For example, it is well known that the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is universal – its shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions.

More interesting and convenient in practical applications is *discrete universality*, when τ takes values from a certain discrete set (e.g., from an arithmetic progression $\{kh : k \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with some fixed $h > 0$). The first discrete universality theorem was proposed by A. Reich and developed by B. Bagchi [1]. One of the most actual problems is to describe discrete sets for which universality theorems are valid. A. Dubickas and A. Laurinčikas [2] obtained a discrete universality theorem for $\zeta(s)$ with shifts from more complicated discrete set, i.e. from the sequence $\{k^\alpha h : k \in \mathbb{N}_0\}$ with a fixed α, $0 < \alpha < 1$. The aim of this talk is to present a generalization of the mentioned A. Dubickas and A. Laurinčikas' result. More precisely, we will discuss the discrete universality of $\zeta(s)$ for the class of sequences $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ satisfying some hypotheses [4]. Also, a new idea on the approximation of a wide class of analytic functions by shifts $\zeta(s + i\gamma_k)$, where $0 < \gamma_1 \leqslant \gamma_2 \leqslant \cdots$ are the imaginary parts of non-trivial zeros of the Riemann zeta-function $\zeta(s)$, will be discussed [3].

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WELL-POSED MATHEMATICAL SIZE-STRUCTURED POPULATION MODEL WITH BOUNDARY TIME DELAY CONDITION

T. V. Pak, S. B. Pak (Far Eastern Federal University, Vladivostok, Russia)

We consider a mathematical model:

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial}{\partial x}[n(x,t)g\ (x)] = -d(x,t;n)n(x,t), \tag{1}$$

where $dx/dt = g(x)$ is the growth rate. Equation (1) is supplemented with the boundary condition

$$R(t) = n(x_0, t)g((x_0) = B(ta)f(B(ta), ta). \tag{2}$$

B(t) is given by

$$B(t) = \begin{cases} B_0(t), 0 \leqslant t \leqslant a, \\ \int_{x_0}^{\infty} p(\xi, t; n)n(\xi, t)\, d\xi, t > a. \end{cases} \tag{3}$$

An ecological content of the model is discussed in [1], [2]. The present paper investigates the existence and uniqueness of the solution of the problem (1) – (4) and it's continuous relationship to boundary and initial conditions. There is equivalent system of characteristic equations for problem (1) – (4)

$$\frac{dt}{1} = \frac{dx}{g(x)} = \frac{dn}{(d(x, t; n) + g'(x))n(x, t)}. \tag{4}$$

Solved this system we move to integral form of our initial task

$$n = S(n), \tag{5}$$

where

$$S(n) = \begin{cases} \frac{R(t-t(\xi))}{g(\xi)} \exp(-\int\limits_{t-t(\xi)}^{t} d(x(\tau, t - t(\xi)), \tau; n)d\tau) & \text{for } \xi \leqslant x(t, a), \\ \frac{\phi(x(\xi,t))g(x(\xi,t))}{g(\xi)} exp(-\int\limits_{a}^{t} d(x(\tau, t - t(\xi)), \tau; n)d\tau & \text{for } \xi > x(t, a). \end{cases} \tag{6}$$

As size of any population is not less than zero and always bounded then $n(x, t)$ – solution of (6) has to satisfy the following requirements :
1) $n(x, t) \geqslant 0$, 2) for $\forall T sup_{t\in[a,T]} \int\limits_{x_0}^{\infty} |n(\xi, t)|d\xi \leqslant +\infty$.

So we can find the solution of the equation (6) in normed space $C(R_1^+ \times [a, T], n(x, t) \in L_1(R_1^+), R_1^+ = [x_0, \infty]$, $C^+(R_1^+) = (z \in C(R_1^+)$ and $z \geqslant 0)$, with bounded norm $\|n\|_C = sup_t\|n(t)\|_L 1, \|n(t)\|_L 1 = \int\limits_{x_0}^{\infty} |n(\xi, t)|d\xi$.

Formalizing of biologically reasonable restrictions on all parameters and functions included into equation (6) we have proved existence of the unique solution of the problem (1) – (4) which continuously relate to boundary and initial conditions. We have found a necessary and sufficient conditions for existence of equilibrium size distribution.

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PROOF OF LÁSZLÓ FEJES TÓTH'S ZONE CONJECTURE

A. A. Polyanskii (Technion — Israel Institute Technology, Haifa; Moscow Institute of of Physics and Technology, Dolgoprudny)

A *zone* of width ω on the unit sphere is the set of points within spherical distance $\omega/2$ of a given great circle. We show that the total width of any collection of zones covering the unit sphere is at least π, answering a question of Fejes Tóth [1] from 1973. Also, we discuss other plank-type problems.

Our talk is based on the joint work [2] with Zilin Jiang.

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A LOGARITHMIC IMPROVEMENT IN THE BOMBIERI-VINOGRADOV THEOREM

A. Sedunova

For integer number a and $q \geqslant 1$, let

$$\psi(x;q,a) = \sum_{\substack{n \leqslant x \\ n \equiv a(\mod q)}} \Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function. The Bombieri-Vinogradov theorem is an estimate for the error terms in the prime number theorem for arithmetic progressions averaged over all q up to $x^{1/2}$, or, rather almost all q up to $x^{\frac{1}{2}}$.

Theorem (Bombieri-Vinogradov). *Let A be a given positive number and $Q \leqslant \frac{x^{1/2}}{(\log x)^B}$, where $B = B(A)$. Then*

$$\sum_{q \leqslant Q} \max_{2 \leqslant y \leqslant x} \max_{\substack{a \\ (a,q)=1}} \left| \psi(y,q,a) - \frac{y}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A}.$$

The implied constant in this theorem is not effective, since we have to take care of characters associated with those q that have small prime factors. At the same time, effective versions – in which the effect of an exceptional character is avoided in one way or another - have been known since the paper by Timofeev and Lenstra-Pomerance work on Gaussian periods.

We improve the best known to date result of Dress-Iwaniec-Tenenbaum, getting $(\log x)^2$ instead of $(\log x)^{\frac{5}{2}}$. We use a weighted form of Vaughan's identity, allowing a smooth truncation inside the procedure, and an estimate due to Barban-Vehov and Graham related to Selberg's sieve. We sketch the proof of effective and non-effective versions of the result. We also briefly discuss the possibility of getting the fully effective Bombieri-Vinogradov theorem for $q \leqslant x^{\frac{1}{2}-\varepsilon}$. The ineffectivity is avoided by applying Landau-Page results without using Siegel-Walfisz theorem.

RECENT RESULTS IN SUM-PRODUCT

I. D. Shkredov (Steklov Mathematical Institute, Moscow, Russia)

We will give a survey on the sum-product phenomenon both in the prime fields and in the real setting. Recent quantitative results connecting with new geometrical observations allow to obtain a series of applications to estimating of exponential sums and to some additive problems.

A WEIGHTED DISCRETE UNIVERSALITY OF PERIODIC ZETA-FUNCTIONS

D. Šiaučiūnas (Šiauliai University, Šiauliai, Lithuania)

The periodic zeta-function $\zeta(s;\mathfrak{a})$, $s = \sigma + it$, where $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ is a periodic sequence of complex numbers, in the half plane $\sigma > 1$ is defined by the series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

Moreover, $\zeta(s;\mathfrak{a})$ can be meromorphically continued to the whole complex plane.

Universality of periodic zeta-functions was considered by various authors: Bagchi, Steuding, Kaczorowski, Laurinčikas, Stoncelis, Šiaučiūnas. Our report is devoted to a weighted discrete universality theorem for the function $\zeta(s;\mathfrak{a})$. Its statement include the weight function $w(u)$, $u \geqslant 1$. Let $V_N = \sum_{k=1}^{N} w(k)$. We suppose that $\lim_{N\to\infty} V_N = +\infty$, moreover, the function $w(u)$ has a continuous derivative satisfying the estimate $\int_1^N u|w'(u)|\mathrm{d}u \ll V_N$. As usual, let $\mathcal{K}$ be the class of compact subsets of the strip $\left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ with connected complements, and $H_0(K)$, $K \in \mathcal{K}$, be the class of continuous non-vanishing functions on K that are analytic in the interior of K. Moreover, let I_A denote the indicator function of the set $A \subset \mathbb{R}$. Then the following theorem is true.

Theorem 1. *Suppose that the function $w(u)$ satisfies the above hypotheses, the sequence $\mathfrak{a}$ is multiplicative, and that α, $0 < \alpha < 1$, and $h > 0$ are fixed. Let*

$K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{V_N} \sum_{k=1}^{N} w(k) I_{\left\{k: \sup\limits_{s\in K} |\zeta(s+ik^\alpha h;\mathfrak{a}) - f(s)| < \varepsilon\right\}}(k) > 0.$$

A continuous version of Theorem 1 was obtained in [1].

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THEORETICAL ANALYSIS OF 2D STATIC MAGNETIC CLOAKING PROBLEMS

Yu. E. Spivak (Far Eastern Federal University, Vladivostok, Russia)

Recently, the use of transformation optics method to design materials technologies that control the propagation of magnetostatic and electrostatic fields has been extensively discussed [1]. This procedure starts with the pioneering study of invisibility devices for material body cloaking from electromagnetic detection by Pendry [2]. Moreover, one can obtain more certain theoretical results when solving cloaking problems for static fields than for the case of electromagnetic waves. This idea was soon confirmed in detail by a few works, for example, [3] for the case of 2D static magnetic cloaking design problem. The solution obtained in [3] suffered from a shortcoming with singularities of magnetic permeabilities at the inner boundary. It is understandable that technical realization of these solutions is impossible as materials corresponding to found solutions do not exist in nature. One of approaches of overcoming these difficulties consists of using approximate non-singular parameters of cloaks and designing cloaking devices based on these approximations. Another approach is connected with using optimization method of solving inverse problems. This method is based on replacing initial inverse cloaking problem by minimization problem of a suitable tracking-type cost functional which corresponds to inverse problem under study.

In this paper the optimization method is applied for theoretical analysis of

2D static magnetic cloaking problems for the case of inhomogeneous externally applied field. The control problems are formulated. The solvability of direct and control problems for the magnetic scattering model under study is proved. The sufficient conditions which provide local uniqueness and stability of optimal solutions are established. Numerical aspects of applying the optimization approach are also discussed.

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GEOMETRY OPTIMIZATION IN THERMAL CLOAKING PROBLEMS

D. A. Tereshko (Institute of Applied Mathematics FEB RAS, Vladivostok)

Recently, a set of novel devices was proposed for ivisibility cloaking in thermal and electric fields (see, for example, [1]). Unfortunately, in many cases these devices are difficult to implement in practice. They usually require complicated medium with anisotropic and spatially varying physical parameters which is not common in naturally occurring materials.

To overcome these technical difficulties an alternative approach for the cloaking shell design was proposed. It is based on using a multilayered cloak consisting of concentric homogeneous layers. This approach is easier to implement in practical device construction but we need to develop a special method to find the optimal physical and geometrical properties of these layers. For this purpose the cloaking problem can be formulated as a constrained minimization problem in which the layers parameters play a role of the controls (see [2]).

In order to solve this problem we proposed numerical algorithm based on the particle swarm optimization [3]. Proposed algorithm was used to solve 2D thermal cloaking problem with anisotropic materials in [2] where the geometric parameters of the cloaking shell (the number of layers and their thickness) were fixed and the components of the thermal conductivity tensor played the role of controls. But it can be difficult to find natural materials with these optimal physical properties. In current study we apply this algorithm for three-dimensional design of cloaking shells with homogeneous and isotropic layers when the materials of the layers are natural and fixed while the thickness of the layers is chosen. We study algorithm properties and discuss computational results.

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ANALYSIS OF 2D THERMAL CLOAKING PROBLEMS USING OPTIMIZATION METHOD

A. S. Tokhtina, V. S. Dukhno

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A lot of attention has been paid recently to developing technologies for designing special functional devices to control heat fluxes. Designing thermal functional devices necessitates solution of inverse problems for the respective heat transfer model. These problems consist in searching for the parameters of a medium that fills the shell with a prescribed topology. The search is based on additional information about the thermal field that is being created. For example, when designing a cloaking shell, the additional information is the requirement that the external response to the thermal scattering that is created by the cloaked object together with the shell and the temperature gradient inside the shell be zero [1]. However, when designing a thermal concentrator, the additional information is the requirement to focus as much thermal energy as possible inside the thermal shell [2].

Problems of designing functional thermal devices have been considered under some simplifying assumptions in a number of publications (see, for example, [2] and references therein). In the general case the above-mentioned theory is not applicable. Nevertheless, in the general case the above inverse problems can be studied with the help of an optimization method using for example the scheme that was proposed in [3] when investigating electromagnetic and acoustic cloaking problems. Just this scheme is used in this paper for studying inverse problems of heat scattering arising when developing the design technologies of special heat devices such as cloaks, concentrators or thermal-illusion devices. Theoretical and computational aspects are discussed.

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ON THE VALUE SET OF GENERAL POLYNOMIALS OVER ARBITRARY $\mathbb{Z}_m$

A. V. Trepacheva (Southern Federal University, Rostov-on-Don/Taganrog)

Let V_f be a value set of polynomial $f \in \mathbb{Z}_m[x]$. When $m \in \mathbb{N}$ is prime number, V_f is well understood [1], however in case of composite m it is not studied thoroughly. The particular problem statement is: given polynomial f to find lower bound on V_f. This problem has great both theoretical and practical value. Only few attempts [2, 3] were made in order to clarify the problem. In [2] the author proposed lower bound on V_f for f such that $1 \leqslant \mathsf{deg}(f) \leqslant 3$. This bound can be extended for higher degrees, but *its evaluation has worse efficiency than explicit exhaustive construction of* V_f. Also the bound from [2] is *inexact* for degree greater than first and its tightness degrades with degree growth. In [3, 4] the problem was studied for the case of $f(x) \in GR(p^n, m)[x]$, where $GR(p^n, m)$ is a Galois ring of order $p^{n \cdot m}$, p is prime. In this work we propose *more tight* and *effectively computable* than in [2] lower bounds for $|V_f|$ over arbitrary $\mathbb{Z}_m$.

Theorem 1. *Let* $m = \prod_{i=1}^{k} p_i^{r_i}$. *For d-th degree polynomial* $f(x) \in \mathbb{Z}_m[x]$ *holds* $\prod_{i=1}^{k} \delta_i \leqslant V_f$, *where* $\delta_i = \left[\frac{p_i - 1}{d}\right] p_i^{r_i - 1} + 1$ *if* $d < 3 \cdot \sqrt{2 \cdot p_i}$, *else* $\delta_i = 1$. $[t]$ *means integer part of a number* t.

Theorem 2. *Let* $m = \prod_{i=1}^{k} p_i$. *For d-th degree polynomial* $f(x) \in \mathbb{Z}_m[x]$ *there is* $\prod_{i=1}^{k} \delta_i \leqslant V_f$, *where* $\delta_i = [\frac{p_i - 1}{d}] + 1$, *if the number of images* $\overline{b}$ *of polynomial* $\overline{f}_{p_i}(x) = [f(x)]_{p_i} \in \mathbb{Z}_{p_i}[x]$, *such that* $\overline{f}_{p_i}(x) - \overline{b}$ *has no simple roots over* $\mathbb{Z}_{p_i}$, *equals to 1. Otherwise* $\delta_i = \max\{\lceil ([\frac{p_i - 1}{d}] + \frac{2 \cdot (p_i - 1)}{d^2} - (d-1)) \cdot p_i^{n-1} + (d-1) \rceil, 1\}$, *holds.*

Here $[f(x)]_{p_i}$ means coefficientwise reduction modulo p_i of $f(x)$. The proof is based on results from [3] and Chineese reminder theorem.

The work is supported by RFBR grant 15-07-00597a.

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SOLUTION OF BALANCE EQUATIONS AND INVESTIGATION OF POISSON FLOWS IN JACKSON NETWORKS

G. Tsitsiashvili (IAM FEB RAS, Vladivostok, Russia; Far Eastern Federal University, Vladivostok, Russia)

In this report a decomposition of a solution of balance equations for intensities of flows departing from nodes of the Jackson network is constructed.

The system of balance equations plays large role in a formulation and a proof of the Jackson product theorem widely used in the queuing theory. If ergodicity conditions are true then discrete Markov process $(n_1(t), \dots, n_m(t)),\ t \geqslant 0$, describing numbers of customers in the network nodes has limit distribution (independent on initial conditions) represented in the form $\prod_{i=1}^{m} p_i(n_i)$, where $p_i(n_i)$ is limit distribution of customers number in isolated l_i -server queuing system with Poisson input flow with an intensity defined by $\lambda_0, \dots, \lambda_m$ [1].

It is known that the Jackson network is confronted with directed graph G which has edges set corresponding positive elements of the route matrix [2]. Procedures of the decomposition are based on a definition of classes of cyclically equivalent nodes and on a construction of acyclic directed graph consistent with the Jackson network and these classes. Then classes of cyclically equivalent nodes are arranged in the acyclic graph accordingly with maximal ways lengths from the source node to all others nodes classes. An algorithm of maximal ways lengths calculation is represented as an analogy of the Floyd-Steinberg algorithm of minimal ways lengths calculation. A decomposition of a solution of balance equations into sub systems of balance equations corresponding classes of the cyclic equivalence is made and an existence and an uniqueness of their solutions is proved.

A problem of a factorization of the graph G by a relation of its nodes cyclic equivalence is considered. An accelerated algorithm of this problem solution based on sequential introduction of new nodes into directed graph and a recalculation of its equivalence classes and partial order between them is represented.

Sets of independent stationary Poisson flows departing from nodes of Jackson network are enumerated. These sets are defined by non-return sets in the acyclic directed graph which is composed of cyclic equivalence classes of the Jackson network nodes. Special algorithm of an enumeration of non-return nodes sets is constructed. This algorithm is constructed accordingly with partial order of the equivalence classes in the acyclic directed graph of the equivalence classes of nodes.

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ELEMENTARY APPROACH TO SOMOC-4 SEQUENCES

A.V. Ustinov (Institute of Applied Mathematics FEB RAS, Pacific National University, Khabarovsk)

Somos–4 sequences are defined by a fourth-order quadratic recurrence relation of the form

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2.$$

The main property of this sequence is the following *near-addition formula* (or so-called *magic determinant*)

$$\begin{vmatrix} s_{m_1+n_1}s_{m_1-n_1} & s_{m_1+n_2}s_{m_1-n_2} & s_{m_1+n_3}s_{m_1-n_3} \\ s_{m_2+n_1}s_{m_2-n_1} & s_{m_2+n_2}s_{m_2-n_2} & s_{m_2+n_3}s_{m_2-n_3} \\ s_{m_3+n_1}s_{m_3-n_1} & s_{m_3+n_2}s_{m_3-n_2} & s_{m_3+n_3}s_{m_3-n_3} \end{vmatrix} = 0,$$

where m_i, n_i $(i = 1, 2, 3)$ are arbitrary integers or half-integers. Usual proof of this property is based on explicit formula for s_n [Hone and Swart, 2003–2005]

$$s_n = CD^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}},$$

where σ is Weierstrass sigma-function. A. van der Poorten and C. Swart [2006] gave tricky but purely algebraic proof (i.e., without using explicit analytic expressions) of this result.

The talk will be devoted to direct inductive proof of magic determinant property. In particular first integral of Somos–4 sequence arise in a natural way. Hopefully this approach will be suitable for higher–rank quadratic recurrence.

ON AN INEQUALITY OF S.V. BOCHKAREV

I. M. Vasilyev (PDMI RAS, Saint-Petersbourg)

In his article [1] S.V. Bochkarev used the following theorem as a tool for lower estimates of certain types of trigonometric sums.

Theorem. *Let* $\{V_n\}$ *be the de la Vallée-Poussin kernels,* $Q_0 := V_1, Q_n := V_{2^n} - V_{2^{n-1}}$. *For a function* $f \in L^1(\mathbb{T})$ *denote*

$$||f||_D := \sup_I \left(\frac{1}{|I|} \int_I \sum_{2^{-n}<|I|} \left(\int_{\mathbb{T}} f(t) Q_n(x-t) dt \right)^2 dx \right)^{\frac{1}{2}}. \tag{1}$$

Then $C_1||f||_D \leqslant ||f||_{BMO(\mathbb{T})} \leqslant C_2||f||_D$ *for some universal constants* C_1 *and* C_2.

This result shows that the quantity $||f||_D$ can be regarded as an equivalent norm for the classical BMO($\mathbb{T}$) space.

In our talk we will present a generalisation of this theorem for the space BMO($\mathbb{R}^n$). We will also discuss some possible applications of that theorem in finding lower bounds for trigonometric sums. The talk will be (partially) based on the paper [2].

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MULTIPLICATIVE ETA–PRODUCTS IN STRUCTURE THEOREMS

G. V. Voskresenskaya (Samara University, Samara)

The following classical fact is well-known: if $k \geqslant 12,$ k is an even positive integer, we have

$$S_k(\Gamma) = \Delta(z) \cdot M_{k-12}(\Gamma),$$

here $\Delta(z) = \eta^{24}(z)$ is the delta–function, $\Gamma = SL_2(\mathbf{Z})$, $\eta(z)$ is the Dedekind's η-function. This phenomenon is called *the exact cutting.*We investigate the generalization of the fact. The delta-function is one of 28 eta–products with multiplicative Fourier coefficients (so-called multiplicative eta-products). These functions have been discovered in 1985 [3].

Theorem. *Let χ be a quadratic character modulo N such that $\chi(-1) = -1$, k, l are positive integers. Then*

$$S_k(\Gamma_0(N), \chi^k) = f(z) \cdot M_{k-l}(\Gamma_0(N), \chi^{k-l}),$$

where $f(z) \in S_l(\Gamma_0(N), \chi^l)$ iff $f(z)$ is a multiplicative eta-product.

Here χ^l is trivial if l is even. We use the Cohen-Oesterle formula for calculating dimensions of modular forms [2] and the Biagioli formula for calculating order in cusps [1]. In this case the space $S_l(\Gamma_0(N), \chi^l)$ is one-dimensional and generated by $f(z)$.

In our talk we shall also speak about other remarkable properties of multiplicative eta-products.

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ЛЮБАЯ НЕВЫРОЖДЕННАЯ ПОСЛЕДОВАТЕЛЬНОСТЬ SOMOS-6 ЯВЛЯЕТСЯ SOMOS-k ПРИ ВСЕХ $k \geqslant 7$

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Для натурального $k > 1$ последовательность Somos-k удовлетворяет квадратичному рекуррентному соотношению

$$A(n+k)A(n) = \sum_{j=1}^{[k/2]} \alpha_j A(n+k-j)A(n+j)$$

с константами α_j. В работе [1] было доказано, что любая невырожденная последовательность Somos-4 является Somos-k для всех $k > 4$. Опираясь на работы [2] и [3] мы доказываем следующий результат.

Теорема. *Любая невырожденная последовательность Somos-6 является Somos-k при всех $k \geqslant 7$.*

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РАСПРЕДЕЛЕНИЕ АЛГЕБРАИЧЕСКИХ И ЦЕЛЫХ АЛГЕБРАИЧЕСКИХ ЧИСЕЛ НА ДЕЙСТВИТЕЛЬНОЙ ПРЯМОЙ

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За последние 20 лет в работах Хаксли [1] и Бересневича, Велани, Диккинсон, Вогана [2] были получены оценки для количества точек с рациональными координатами в окрестности гладкой кривой.

В докладе будут представлены результаты, связанные с оценками для количества точек с алгебраически сопряженными координатами в коротких интервалах [3], параллелепипедах малой меры и в окрестности гладких кривых. Эти результаты были получены авторами [4] и являются естественным обобщением отмеченных выше работ.

Пусть $I \subset \mathbb{R}$ некоторый интервал, $f : I \to \mathbb{R}$ непрерывно дифференцируемая на I функция. Для натуральных чисел $Q > 1$ и $n \geqslant 2$ определим следующий класс многочленов

$$\mathcal{P}_n(Q) := \{P \in \mathbb{Z}[t] : \deg P \leqslant n, H(P) \leqslant Q\},$$

где $H(P)$ - высота многочлена P. Для действительного $0 < \gamma < 1$ обозначим через $\mathcal{M}_n(Q, \gamma)$ множество точек $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ с действительными алгебраически сопряженными координатами, минимальный многочлен которых принадлежит классу $\mathcal{P}_n(Q)$ и которые удовлетворяют условию

$$|f(\alpha_1) - \alpha_2| < c_1 Q^{-\gamma}.$$

Theorem. *При $Q \to \infty$ справедливо неравенство*

$$c_2 Q^{n+1-\gamma} < \#\mathcal{M}_n(Q, \gamma) < c_3 Q^{n+1-\gamma},$$

где c_1, c_2, c_3 - положительные величины, не зависящие от Q.

Данная теорема может быть рассмотрена в случае целых алгебраических чисел и в случае функций $f : \mathbb{R}^k \to \mathbb{R}$ многих переменных.

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ФУНКЦИОНАЛЬНЫЕ УРАВНЕНИЯ ДЛЯ ТЕТА-ФУНКЦИИ

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Рассмотрим функциональное уравнение

$$f_1(x_1+z)\dots f_s(x_s+z)g(x_1+\dots+x_s-z)=\sum_{j=1}^{m}\phi_j(x_1,\dots,x_s)\psi_j(z). \qquad (1)$$

относительно неизвестных функций $f_l, g, \psi_j : \mathbb{C} \to \mathbb{C}$, $\phi_j : \mathbb{C}^s \to \mathbb{C}$, $l = \overline{1,s}$, $j = \overline{1,m}$. Оно представляет интерес с точки зрения теорем сложения и их приложений к задачам математической физики (см., например, [1, 2]). Общее его решение известно только при $s = 1$, $m \leqslant 2$ (см. [3]), $s = 2$, $m = 3$ (см. [4]), а также $s = 1$, $m = 3$ и $s = 2$, $m = 4$ (см. [5, 6]). В этих случаях все неэлементарные решения выражаются через тета-функцию Якоби. В докладе пойдет речь об этих результатах, а также их обобщений на случай произвольного s. В частности, мы доказываем что любой набор неэлементарных функций $(f_1,\dots,f_s,g)$, удовлетворющий вместе с некоторыми ϕ_j, ψ_j разложению (1) при $m \leqslant s+2$ имеет (с точностью до некоторых простых преобразований) вид

$$(f_1, \dots, f_s, g) = (\theta, \dots, \theta, \theta),$$

где θ — некоторая тета-функция Якоби. Таким образом, функциональное уравнение (1) можно рассматривать как «характеристическое» соотношение для тета-функции при $m = s + 1$.

Исследование выполнено за счет гранта Российского научного фонда (проект N 14-11-00335).

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ЧИСЛЕННОЕ РЕШЕНИЕ ОПТИМИЗАЦИОННОЙ ЗАДАЧИ С ОГРАНИЧЕНИЯМИ ДЛЯ ПРОЦЕССОВ ДИФРАКЦИИ АКУСТИЧЕСКИХ ВОЛН

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Работа посвящена исследованию оптимизационной задачи дифракции стационарных акустических волн в однородной среде с включением. Такие задачи встречаются в дефектоскопии, акустике океана и атмосферы, геофизике. Данная работа является продолжением статей [1, 2], посвященных применению метода граничных интегральных уравнений для численного решения прямых и оптимизационных задач дифракции.

Пусть в пространстве R^3, заполненном однородной изотропной средой, имеется однородное ограниченное изотропное включение Ω_i со связной границей S. Положим $\Omega_e = R^3 \backslash \overline{\Omega}_i$. Предположим, что в области Ω_e имеются источники звука. Звуковые волны распространяются в пространстве и, достигая включения, рассеиваются на нем. В результате, в области Ω_e возникают отраженные волны, а в Ω_i — проходящие волны.

Рассмотрим следующую задачу: изменяя источники звука в Ω_e минимизировать отклонение поля давлений в Ω_i (либо на некотором подмножестве $Q \subset \Omega_i$) от некоторого требуемого.

В [2] был предложен и реализован на ЭВМ алгоритм численного решения задачи для случая, когда ограничения на управление отсутствуют. Там же доказана корректность задачи и сходимость разработанного алгоритма. В настоящей работе рассматривается случай, когда мощность источников, с помощью которых мы можем управлять акустическим полем, ограничена. Проведены тестовые расчеты и численные эксперименты.

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ОЦЕНКИ КОЛИЧЕСТВА ДЕЙСТВИТЕЛЬНЫХ АЛГЕБРАИЧЕСКИХ ЧИСЕЛ В КОРОТКИХ ИНТЕРВАЛАХ

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Зафиксируем натуральное число $n \geqslant 1$, достаточно большое число $Q \in \mathrm{N}$ и конечный интервал $I \subset [0;1)$ длины $\mu I = Q^{-\gamma}$, $0 \leqslant \gamma \leqslant 1$. Обозначим через $\mathrm{M_I}(n, Q)$ множество действительных алгебраических чисел α степени n, высоты $H(\alpha) \leqslant Q$, лежащих в интервале I.

Задача состоит в получении асимптотических при $Q \to \infty$ оценок для количества $\#\mathrm{M}_{\mathrm{I}}(n, Q)$ алгебраических точек. В работе [1] доказано, что существуют величины $c_1(n)$, $c_2(n)$, $c_3(n)$ и интервалы I длины $c_1(n)Q^{-1}$ такие, что $\#\mathrm{M}_{\mathrm{I}}(n, Q) = \varnothing$. Однако если величина $c_2(n)$ достаточно большая, то при $0 \leqslant \mu \leqslant 1$ имеем $\#\mathrm{M}_{\mathrm{I}}(n, Q) > c_3(n)Q^{n+1}\mu I$.

Теорема. *Для любого $\varepsilon > 0$ и интервала длины $Q^{-\gamma}$, $0 \leqslant \gamma < 1$ можно найти величины $c_3(n)$, $c_4(n)$, при которых справедливо неравенство*

$$c_3(n)Q^{n+1-\frac{\gamma}{2}}\mu I < \#M_I(n, Q) < c_4(n)Q^{n+1+\varepsilon}\mu I. \qquad (1)$$

Доказательство (1) основывается на эффективных теоремах метрической теории диофантовых приближений, начало которых было положено в работах [2], [3].

При $\gamma > 1$ можно также получить оценки вида (1), однако при этом хуже становится зависимость от γ и на интервал I надо наложить дополнительные условия, потребовав отсутствие на I действительных алгебраических чисел малой степени и высоты. Результаты могут быть обобщены с интервала I на круги K в комплексной плоскости радиуса $r = Q^{-\gamma_1}$, $0 \leqslant \gamma_1 < 1$ [4].

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О МУЛЬТИПЛИКАТИВНОМ МЕТОДЕ ПОСТРОЕНИЯ ЦЕЛОЧИСЛЕННЫХ ПОСЛЕДОВАТЕЛЬНОСТЕЙ СОМОС-8 И СОМОС-9

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Пусть k — любое натуральное число большее единицы и $\alpha_1, \ldots, \alpha_{[k/2]}$ – произвольный набор из $[k/2]$ комплексных чисел, одновременно не равных нулю. Последовательность $A : \mathbb{Z} \to \mathbb{C}$, удовлетворяющая квадратичному рекуррентному соотношению

$$A(n+k)A(n) = \sum_{i=1}^{[k/2]} \alpha_i A(n+k-i)A(n+i) \qquad (n \in \mathbb{Z}) \tag{1}$$

называют Сомос-k.

Выберем вместо набора комплексных чисел $(\alpha_1, \ldots, \alpha_{[k/2]})$ набор из $[k/2]$ формальных переменных $u = (u_1, \ldots, u_{[k/2]})$, а вместо набора $(A(1), \ldots, A(k))$ из k комплексных чисел набор $X = (X_1, \ldots, X_k)$ с формальными переменными X_i. Построим последовательность рациональных функций

$$Q_k(n) = Q_k(u; X; n) = Q_k(u_1, \ldots, u_{[k/2]}; X_1, \ldots, X_k; n),$$

положив в качестве k начальных значений

$$Q_k(1) = X_1, \ldots, Q_k(k) = X_k.$$

Далее продолжаем ею вправо $(n > 0)$ и влево $(n \leqslant 0)$ по рекуррентным формулам

$$Q_k(n+k) = \frac{1}{Q_k(n)} \sum_{i=1}^{[k/2]} u_i Q_k(n+k-i) Q_k(n+i),$$

$$Q_k(n) = \frac{1}{Q_k(n+k)} \sum_{i=1}^{[k/2]} u_i Q_k(n+k-i) Q_k(n+i).$$

В работах М. Уорда и М. Сомоса начал изучаться вопрос о том, при каких условиях наложенных на коэффициенты α_i и начальные значения $A(j)$ все элементы последовательности Сомос–k являются целочисленными для $k \geqslant 4$. При $k = 2, 3$, ввиду того, что существуют явные формулы для $Q_k(n)$, ответ очевиден.

В работе [1] было доказано, что при $4 \leqslant k \leqslant 7$

$$Q_k(u_1, \dots, u_{[k/2]}; X_1, \dots, X_k; n) = \sum_{l=(l_1,\dots,l_k)\in\mathbb{Z}^k} P_l(u_1, \dots, u_{[k/2]}) X_1^{l_1}(1) \dots X_k^{l_k}(k),$$

где P_l — полином от переменных u_i с целыми коэффициентами. Отсюда следует, что если $\alpha_1, \dots, \alpha_{[k/2]}$ — целые числа, а начальные значения $A(1), \dots, A(k)$ равны ± 1, то A — целочисленная последовательность Сомос–k. В работе [2], опирающейся на [3], для $k = 4, 5$ были получены более сильные результаты.

Теорема. *Пусть A, B — целочисленные последовательности Сомос-4. Тогда последовательность $C : \mathbb{Z} \to \mathbb{Z}$, определяемая по формуле*

$$C(n) = A(n)B(n), \tag{2}$$

— целочисленная последовательность Сомос-k, где $k \leqslant 9$.

Работа поддержана грантом РНФ (проект №14-11-00335).

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ПРИМЕНЕНИЕ ОТНОСИТЕЛЬНОЙ ЕМКОСТИ К ОЦЕНКЕ ПРОИЗВОДНОЙ ШВАРЦА ГОЛОМОРФНЫХ ФУНКЦИЙ

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Рассмотрим класс B функций f голоморфных в единичном круге $U_z = \{z : |z| < 1\}$ с разложением: $f(z) = 1 + a_1(z-1) + a_2(z-1)^2 + a_3(z-1)^3 + \angle o((z-1)^3)$. С помощью понятия относительной емкости [1] методами теории потенциала [2] получен следующий результат.

Теорема. *Пусть функция f из класса B, и пусть угловая мера Лебега пересечения образа единичного круга $f(U_z)$ с каждым $r = |w|$, $0 < r_0 < r < 1$, меньше либо равна α, $0 < \alpha < 2\pi$, тогда справедлива оценка:*

$$-\frac{1}{6}\operatorname{Re}\frac{S_f(1)}{(f'(1))^2} \leqslant$$

$$\frac{1}{12} - \operatorname{Re}[\frac{1}{\pi^2\alpha^2}(K^2(6k^2g(1)^2 - k^2 - 1) - \frac{3K^2g(1)^2(2k^2g(1)^2 - k^2 - 1)^2}{2(1-g(1)^2)(1-k^2g(1)^2)})].$$

Равенство достигается в случае функции

$$f_\alpha(z) = \exp(\frac{i\pi\alpha}{K}u(g(z), k)),$$

конформно и однолистно отображающей единичный круг U_z на множество $B(\alpha, r_0) = \{z :\, r_0 < |z| < 1,\, |argz| < \alpha/2\}$.

Здесь

$$u(g(z), k) = \int\limits_0^{g(z)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},\ 0 < k < 1,$$

$$g(z) = e^{i\theta}\frac{z-\beta}{z-\overline{\beta}},\ \ \forall\theta, \forall\beta,\ \ 0 \leqslant \theta < 2\pi,\ \operatorname{Im}\beta > 0,$$

$$K = u(1, k),\ K' = u(1, \sqrt{1-k^2}),\ \log r_0 = -K'\frac{\pi\alpha}{K}.$$

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Оглавление