INTERNATIONAL CONFERENCE

BAIKAL NUMBER THEORY

ABSTRACTS

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- Institute for Applied Mathematics, Far-Eastern Branch of the Russian Academy of Sciences (IAM FEB RAS)
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- Department of Mechanics and Mathematics, Moscow State University (MSU)

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AUTHOR'S INDICATOR

Abstracts of reports

THE ESTIMATION OF THE DISPERSION FOR THE LENGTH OF CONTINUED FRACTIONS OF RATIONAL NUMBERS WITH PRIME NUMERATORS

Mariia O. Avdeeva (Khabarovsk division IAM FEB RAS, Khabarovsk, Russia)

For rational r denote by $s(r)$ the length of continued fraction expansion $r = [q_0; q_1, \ldots, q_s]$. It is know that average value of $s(a/d)$ $(1 \leq a \leq d)$ is $\frac{12\log 2}{\pi^2}\log d$ and (see [1])

$$\frac{1}{d}\sum_{a=1}^{d}\left(s\left(\frac{a}{d}\right) - \frac{12\log 2}{\pi^2}\log d\right)^2 \ll \log d$$

The talk will be denoted to the following result.

Theorem. *For any* $d > 10$

$$\frac{\log d}{d}\sum_{\substack{1<p\leq d\\ p \text{ is prime}}}\left(s\left(\frac{p}{d}\right) - \frac{12\log 2}{\pi^2}\log d\right)^2 \ll \frac{\log^2 d}{\log\log d}$$

This research is supported by the Russian Science Foundation (project 19-11-00065).

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ON SOME GENERALIZED CATALAN NUMBERS

Christian Ballot (Université de Caen, Caen, France)

If $A = (a_n)_{n\geq 0}$ is a sequence of nonzero integers, then one may consider the generalized binomial coefficients, $\binom{m}{n}_A$, with respect to A. They are defined for $m \geq n \geq 0$ as follows

$$\binom{m}{n}_A = \frac{a_m a_{m-1} \ldots a_{m-n+1}}{a_n a_{n-1} \ldots a_1},$$

if $m \geq n \geq 1$, and as 1, if $n = 0$.

When A is a fundamental Lucas sequence, i.e., a second-order integral linear recurrent sequence with initial values $a_0 = 0$ and $a_1 = 1$, then these generalized binomials are called *Lucasnomials*. They are always integers.

We know $n+1$ divides the middle binomial coefficient $\binom{2n}{n}$ for all $n \geq 1$. In fact,

$$C_n := \frac{1}{n+1}\binom{2n}{n},$$

is the famous and seemingly ubiquitous sequence of Catalan numbers. However, if k is an integer not 1, then there are infinitely many $n \geq 1$ for which $n+k$ does not divide $\binom{2n}{n}$. As it happens this Catalan phenomenon remains true, or nearly so, for Lucasnomials, i.e., except for one particular pair (U, k), given $k \neq 1$, there are infinitely many n such that U_{n+k} does not divide $\binom{m}{n}_U$, where U is a fundamental Lucas sequence.

We will go in detail into theorems surrounding this phenomenon, indicate further generalizations and open problems. The mathematics involved are all elementary and this talk will be accessible to a broad audience.

UNIVERSAL FORMAL GROUP FOR ELLIPTIC GENUS OF LEVEL N

Elena Yu. Bunkova (Steklov Mathematical Institute RAS, Moscow, Russia)

The elliptic function of level N determines the elliptic genus of level N as a Hirzebruch genus. It is known that the elliptic function of level N is a specialization of the Krichever function that determines the Krichever genus. The Krichever function is the exponential of the universal Buchstaber formal group.

In the talk we present a specialization of the Buchstaber formal group, such that it determines formal groups that correspond to the elliptic genus of level N:

The elliptic function of level N is the exponential of the formal group of the form

$$F(u,v) = \frac{u^2 A(v) - v^2 A(u)}{uB(v) - vB(u)},$$

where $A(u), B(u) \in \mathbb{C}[[u]]$, $A(0) = B(0) = 1$, $A''(0) = B'(0) = 0$, and for $n = \left[\frac{N-1}{2}\right]$, $m = \left[\frac{N-2}{2}\right]$ there exist parameters $(a_1, \ldots, a_m, b_1, \ldots, b_n)$, such that the relation holds

$$(B(u)+b_1u)^2(B(u)+b_2u)^2 \ldots (B(u)+b_{n-1}u)^2(B(u)+b_nu)^{N-2n} = \\ = A(u)^2(A(u)+a_1u^2)^2 \ldots (A(u)+a_{m-1}u^2)^2(A(u)+a_mu^2)^{N-1-2m}.$$

For the universal formal group of this form the exponential is an elliptic function of level not greater than N.

This proposition is a generalization to $N > 2$ of a known result, that the elliptic function of level 2, determining the elliptic Ochanine–Witten genus, is the exponential of the universal formal group of the form

$$F(u,v) = \frac{u^2 - v^2}{uB(v) - vB(u)},$$

where $B(u) \in \mathbb{C}[[u]]$, $B(0) = 1$, $B'(0) = 0$.

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СПЕКТРАЛЬНЫЕ РАЗЛОЖЕНИЯ СРЕДНИХ ПО ЦЕЛЫМ ТОЧКАМ НА ГИПЕРБОЛОИДАХ И ИХ ПРИЛОЖЕНИЯ

Viktor A. Bykovskii (Khabarovsk division IAM FEB RAS, Khabarovsk, Russia)

В докладе будет рассказано о последних результатах автора и других специалистов по теории чисел, полученных спектральными методами теории автоморфных функций.

Работа выполнена при поддержке гранта РНФ (№19-11-00065).

DIRICHLET SERIES ALGEBRA OF A MONOID OF NATURAL NUMBERS

Nikolai Dobrovol'skii (Tula State University,
Tula State L. N. Tolstoy Pedagogical University, Tula, Russia)

For an arbitrary monoid of natural numbers, the foundations of the Dirichlet series algebra are constructed either over a numerical field or over a ring of integers of an algebraic numerical field.

For any numerical field $\mathbb{K}$, it is shown that the set $\mathbb{D}^*(M)_{\mathbb{K}}$ of all reversible Dirichlet series of $\mathbb{D}(M)_{\mathbb{K}}$ is an infinite Abelian group consisting of series whose first coefficient is nonzero.

We introduce the notion of an integer Dirichlet monoid of natural numbers that form an algebra over a ring of algebraic integers $\mathbb{Z}_{\mathbb{K}}$ of the algebraic field $\mathbb{K}$. It is shown that for a group $\mathbb{U}_{\mathbb{K}}$ of algebraic units of the ring of algebraic integers of $\mathbb{Z}_{\mathbb{K}}$ an algebraic field $\mathbb{K}$ the set of $\mathbb{D}(M)_{\mathbb{U}_{\mathbb{K}}}$ of entire Dirichlet series, $a(1) \in \mathbb{U}_{\mathbb{K}}$, is multiplicative group.

For any Dirichlet series from the Dirichlet series algebra of a monoid of natural numbers, the reduced series, the irreversible part and the additional series are determined. A formula for decomposition of an arbitrary Dirichlet series into the product of the reduced series and a construction of an irreversible part and an additional series is found.

For any monoid of natural numbers allocated to the algebra of Dirichlet series, convergent in the entire complex domain. The Dirichlet series algebra with a given half-plane of absolute convergence is also constructed. It is shown that for any nontrivial monoid M and for any real σ_0, there is an infinite set of Dirichlet series of $\mathbb{D}(M)$ such that the domain of their holomorphism is α-half-plane $\sigma > \sigma_0$.

With the help of the universality theorem S. M. Voronin managed to prove the weak form of the universality theorem for a wide class of Zeta functions of monoids of natural numbers.

In conclusion describes the actual problem with the Zeta functions of monoids of natural numbers that require further research. In particular, if the Linnik–Ibrahimov hypothesis is true, then a strong theorem of universality should be valid for them.

Acknowledgments: The reported study was funded by RFBR, project number 19-41-710004_r_a.

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SUMS OF RECIPROCALS OF FRACTIONAL PARTS

Reynold Fregoli (Royal Holloway (University of London), Egham, UK)

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $Q \geq 1$. We set

$$S_\alpha(Q) := \sum_{q=1}^{Q} \|\alpha q\|^{-1},$$

where $\|\cdot\|$ denotes the distance from the nearest integer. Sharp asymptotic bounds are known for the sum $S_\alpha(Q)$ in terms of the Diophantine type of α. In a 2013 paper, Lê and Vaaler consider a generalisation of $S_\alpha(Q)$, where α is replaced by a matrix. Lê and

Vaaler provide general lower bounds and conditional upper bounds for these generalised sums. In particular, their upper bounds hold only if the matrix provides a counterexample to the Littlewood conjecture. In this talk we show that this condition can be relaxed, and that indeed Lê and Vaaler's upper bounds hold for a much broader class of matrices. This is a consequence of a general counting principle for weakly admissible lattices.

BOUNDS ON LINEAR FORMS AND INTEGER SPARSE RECOVERY

Lenny Fukshansky (Claremont McKenna College, Claremont, USA))

We investigate the problem of constructing m by d matrices A with small entries and d large comparing to m so that for all integer vectors x in R^d with at most m nonzero coordinates the Euclidean norm of the image vector Ax is bounded away from 0 by an absolute constant. A bound like this allows for robust recovery of the original sparse vector x from its image Ax. This problem is motivated by the compressed sensing paradigm and has numerous potential applications ranging from wireless communications to medical imaging. We use a combination of combinatorial, probabilistic and number-theoretic methods to discuss existence and some constructions of such sensing matrices with concrete examples. We also discuss limitations of our constructions, stemming from sparse variations of some classical results in the geometry of numbers. This is joint work with D. Needell and B. Sudakov.

TRANSFERENCE PHENOMENON IN THE WEIGHTED SETTING

Oleg N. German (Pacific National University, Khabarovsk, Russia)

The talk is devoted to generalising the existing transference inequalities for regular and uniform Diophantine exponents to the weighted setting.

Let us fix weights

$$\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_m)\in\mathbb{R}^m_{>0},\qquad \boldsymbol{\rho}=(\rho_1,\ldots,\rho_n)\in\mathbb{R}^n_{>0},$$

$$\sigma_1\geqslant\ldots\geqslant\sigma_m,\qquad \rho_1\geqslant\ldots\geqslant\rho_n,\qquad \sum_{j=1}^m\sigma_j=\sum_{i=1}^n\rho_i=1,$$

and define the weighted norms $|\cdot|_{\boldsymbol{\sigma}}$ and $|\cdot|_{\boldsymbol{\rho}}$ by

$$|\mathbf{x}|_{\boldsymbol{\sigma}}=\max_{1\leqslant j\leqslant m}|x_j|^{1/\sigma_j}\qquad\text{for }\mathbf{x}=(x_1,\ldots,x_m),$$

$$|\mathbf{y}|_{\boldsymbol{\rho}}=\max_{1\leqslant i\leqslant n}|y_i|^{1/\rho_i}\qquad\text{for }\mathbf{y}=(y_1,\ldots,y_n).$$

Consider the system of inequalities

$$\begin{cases}|\mathbf{x}|_{\boldsymbol{\sigma}}\leqslant t\\ |\Theta\mathbf{x}-\mathbf{y}|_{\boldsymbol{\rho}}\leqslant t^{-\gamma}\end{cases}.\tag{1}$$

Definition 1. The *weighted Diophantine exponent* $\omega_{\boldsymbol{\sigma},\boldsymbol{\rho}}(\Theta)$ is defined as the supremum of real γ such that the system (1) admits nonzero solutions in $(\mathbf{x},\mathbf{y})\in\mathbb{Z}^{m+n}$ for some arbitrarily large t.

Definition 2. The *uniform weighted Diophantine exponent* $\hat{\omega}_{\boldsymbol{\sigma},\boldsymbol{\rho}}(\Theta)$ is defined as the supremum of real γ such that the system (1) admits nonzero solutions in $(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^{m+n}$ for every t large enough.

We present the following two theorems, which generalise to the weighted case Dyson's inequality for regular exponents and the author's inequalities for uniform exponents. Let $\Theta^\top$ denote the transpose of Θ.

Theorem 1. *Set* $\omega = \omega_{\boldsymbol{\sigma},\boldsymbol{\rho}}(\Theta)$ *and* $\omega^\top = \omega_{\boldsymbol{\rho},\boldsymbol{\sigma}}(\Theta^\top)$. *Then*

$$\omega^\top \geqslant \frac{\left(\rho_n^{-1} - 1\right) + \sigma_m^{-1}\omega}{\rho_n^{-1} + \left(\sigma_m^{-1} - 1\right)\omega}.$$

Theorem 2. *Set* $\hat{\omega} = \hat{\omega}_{\boldsymbol{\sigma},\boldsymbol{\rho}}(\Theta)$ *and* $\hat{\omega}^\top = \hat{\omega}_{\boldsymbol{\rho},\boldsymbol{\sigma}}(\Theta^\top)$. *Then*

$$\hat{\omega}^\top \geqslant \begin{cases} \dfrac{1-\sigma_m\hat{\omega}^{-1}}{1-\sigma_m} & \text{if } \hat{\omega} \geqslant \sigma_m/\rho_n \\[2ex] \dfrac{1-\rho_n}{1-\rho_n\hat{\omega}} & \text{if } \hat{\omega} \leqslant \sigma_m/\rho_n \end{cases}.$$

This research is supported by the Russian Science Foundation (project 18-41-05001).

TRIANGULAR-LIKE NUMBERS WHICH ARE TRIANGULAR

Gopal Krishna Panda and Sushree Sangeeta Pradhan

(National Institute of Technology Rourkela, Odisha, India)

A balancing-like sequence is a recurrence sequence satisfying the recurrence relation $x_{n+1} = Ax_n - x_{n-1}$ with initial terms $x_0 = 0$ and $x_1 = 1$ and $A > 2$ is a positive integer. Such a sequence is denoted by $BL(A,-1)$. For any given A, the n-th triangular-like number is defined as $\tau_n(A) = \frac{x_n \cdot x_{n+1}}{A}$. All the triangular-like numbers corresponding to the balancing-like sequence with $A = 4$ are triangular numbers. However, no other balancing-like sequence enjoys this property. Furthermore, the N-th triangular-like number of the sequence $BL(A,-1)$ is equal to the sum of first n terms of the sequence $BL(A^2-2,-1)$.

NEW FORMULAS FOR VIETA–JACOBSTHAL AND VIETA–JACOBSTHAL–LUCAS POLYNOMIALS

Taras Goy (Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine)

In [3], authors defined Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials by the recurrences $G_n(x)=G_{n-1}(x)-2xG_{n-2}(x)$ and $g_n(x)=g_{n-1}(x)-2xg_{n-2}(x)$, respectively, where $G_0(x)=0$, $G_1(x)=1$, $g_0(x)=2$, $g_1(x)=1$. Note that $G_n(-\frac{1}{2})=F_n$ and $g_n(-\frac{1}{2})=L_n$, where F_n and L_n are the nth Fibonacci and Lucas number.

We obtained some identities involving polynomials $G_n(x)$ and $g_n(x)$ and multinomial coefficients. Our approach is similar in spirit to [1, 2].

Let $|t|=t_1+\cdots+t_n$, $\tau_n=t_1+2t_2+\cdots+nt_n$, and $p_n(t)=\frac{(-1)^{|t|}|t|!}{t_1!\cdots t_n!}$.

Theorem. *Let $n\geq 2$, except then noted otherwise. Then*

$$\sum_{\tau_n=n} p_n(t)G_2^{t_1}(x)G_3^{t_2}(x)\cdots G_{n+1}^{t_n}(x)=0, \qquad n\geq 3,$$

$$\sum_{\tau_n=n} p_n(t)G_3^{t_1}(x)G_4^{t_2}(x)\cdots G_{n+2}^{t_n}(x)=(2x)^n,$$

$$\sum_{\tau_n=n} p_n(t)G_3^{t_1}(x)G_5^{t_2}(x)\cdots G_{2n+1}^{t_n}(x)=(-2x)^{n-1},$$

$$\sum_{\tau_n=n} p_n(t) g_2^{t_1}(x) g_3^{t_2}(x) \cdots g_{n+1}^{t_n}(x) = 2^{2n-3}(8x+1)x^{n-1},$$

$$\sum_{\tau_n=n} p_n(t) g_3^{t_1}(x) g_4^{t_2}(x) \cdots g_{n+2}^{t_n}(x) = 2(2x)^{n-1}\big((2^{n+1}-1)x + 2^{n-2}\big),$$

$$\sum_{\tau_n=n} p_n(t) g_3^{t_1}(x) g_5^{t_2}(x) \cdots g_{2n+1}^{t_n}(x) = (8x+1)(2x)^{n-1},$$

where the summation is over integers $t_i \geq 0$ satisfying $\tau_n = n$.

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POSITIVITY OF CHARACTER SUM

Alexander B. Kalmynin (Pacific National University, Khabarovsk, Russia)

Let α be a positive real number. For any prime p let $L(\alpha, p)$ be the sum of first $\alpha * p$ Legendre symbols modulo p. It turns out that this quantity is in some sense positively biased. We will discuss several results concerning positivity of $L(\alpha, p)$ and connections between $L(\alpha, p)$ and random multiplicative functions.

This research is supported by the Russian Science Foundation (project 18-41-05001).

GENERALIZED CONTINUED FRACTIONS OF BERNOULLI AND RELATED NUMBERS, AND THEIR APPLICATIONS

Takao Komatsu (Department of Mathematical Sciences, School of Science, Hangzhou, China)

In this talk, we give several new continued fraction expansions of Bernoulli, Cauchy, Euler and related numbers. We also show some applications of generalized continued fraction expansions to caterpillar trees in graph theory.

THE DISTRIBUTION OF THE RATIONAL POINTS ON THE UNIT CIRCLE

Maxim A. Korolev (Steklov Mathematical Institute, Russian Academy of Science, Moscow, Russia), ***Alexey V. Ustinov*** (Pacific National University, Khabarovsk, Russia)

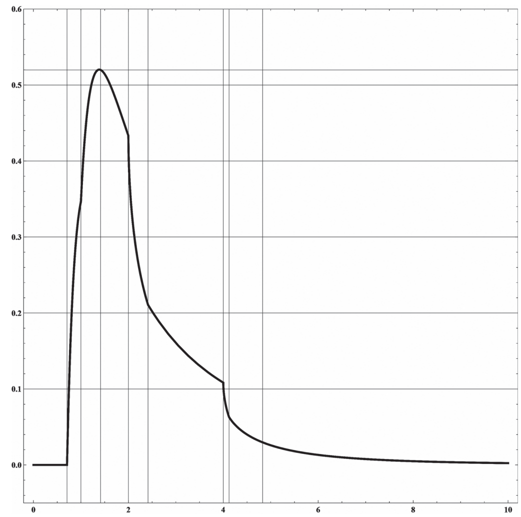
We give an explicit form of distribution function for the normalized lengths of arcs connecting neighbouring rational points on the unit circle whose denominators do not exceed given value.

Suppose that $Q \geq 2$ and let (x_j, y_j), $j = 1, 2, \ldots, N$ ($N = N(Q) \sim Q/\pi$), be all the points on the unit circle whose coordinates are positive irreducible fractions with the denominators not exceeding Q and ordered according to their polar angles $\varphi_j = \operatorname{arctg}(y_j/x_j)$. Further, let $\theta_j = \varphi_j - \varphi_{j-1}$ ($2 \leq j \leq N$) be gaps between neighbouring polar angles. Finally, suppose that $t > 0$ is any fixed positive value and denote by $N(Q;t)$ the number of gaps such that $\theta_j \leq \frac{t}{Q}$.

Theorem.

$$N(Q;t) = N(Q)\int_0^t h(v)dv + O\big(t^{1/2}Q^{5/6}(\log Q)^{4/3}\big),$$

where h is some explicit function.



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CHARACTERIZATION OF 2-PISOT SERIES IN $q = 2^r$ CHARACTER

Hassen Kthiri (Faculty of science of sfax, Department of Mathematics, University of Sfax, Tunisia)

Let $q = 2^r$ and $\mathbb{F}_q((X^{-1}))$ be the field of formal power series over a finite field $\mathbb{F}_q$. In this work, we characterize a pair of roots that lies outside the unit disc while all remaining conjugates have a modulus strictly less than 1. In particular, we provide a sufficient condition for a pair of formal power in $(\mathbb{F}_q((X^{-1})))^2$ series to be a 2-Pisot series. We also give an irreducibility criterion over $\mathbb{F}_q[X]$.

Similarly to the real case, we define 2-Pisot numbers as a pair $(w_1, w_2) \in (\mathbb{F}_q((X^1)))^2$ of conjugate algebraic integers over $\mathbb{F}_q[X]$

such that $|w_1| > 1$ and $|w_2| > 1$, whose remaining conjugates in $\overline{\mathbb{F}_q((X^{-1}))}$ have an absolute value strictly smaller than 1. We denote by S^* the set of all 2-Pisot numbers.

ON PAIR CORRELATION OF SEQUENCES

Gerhard Larcher (JKU Linz, Austria)

The introduction and the analysis of the concept of pair correlation of sequences in the unit-interval was motivated by certain conjectures in quantum physics, and was started – from the mathematical point of view – in several papers by Rudnick, Sarnak and Zaharescu in the 1990's. The pair correlation of a sequence in some sense measures the distribution of "small distances" of points of the sequence. In the last years there grew new interest in this topic by several research papers in which a strong connection of the concept of pair correlation to questions and concepts from additive combinatorics and from uniform distribution theory was discovered. In this talk we give a survey on the concept, on basic and on new properties and results, and we state several open problems. Further we give a multi-dimensional concept of pair correlation and its basic properties.

АСИМПТОТИЧЕСКАЯ ФОРМУЛА ДЛЯ СВЕРТКИ ФУНКЦИИ ЧИСЛА ПРЕДСТАВЛЕНИЙ НАТУРАЛЬНЫХ ЧИСЕЛ СУММОЙ ДВУХ КВАДРАТОВ

Mariia D. Monina (Khabarovsk division IAM FEB RAS, Khabarovsk, Russia)

Пусть d — неотрицательное целое число и $R(d)$ — количество представлений d в виде суммы четырёх квадратов целых чисел. В 1829 году Якоби доказал, что для нечетного натурального d

$$R(d) = 8\sigma(d) = 8\sum_{d_1|d} d_1.$$

С помощью функции $r(n)$ — числа представлений неотрицательного целого n, это утверждение можно записать в следующем виде

$$R(d) = 8\sum_{0\le n\le d} r(d)r(d-n) = 8\sigma(d).$$

В докладе будет представлен следующий результат.

Theorem. *Пусть $-1 < \alpha < \beta < 1$. Тогда для любого нечётного натурального d и любого $\varepsilon > 0$ равномерно по α и β*

$$\sum_{\alpha d<n<\beta d} r(n)r(d-n) = 4(\beta-\alpha)\sigma(d) + O_\varepsilon(d^{1/2+\varepsilon}).$$

Доказательство опирается на формулы следа из работы [1] и теорему разложения функций по ортогональной системе полиномов Лежандра (см. [2]).

Работа выполнена при поддержке гранта РНФ (№ 19-11-00065).

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IRRATIONALITY MEASURE FUNCTIONS

Nikolay Moshchevitin (Pacific National University, Khabarovsk, Russia)

We study the ordinary irrationality measure function

$$\psi_\alpha(t) = \min_{q\leq t} ||q\alpha||,$$

as well as some other ones, such as Minkowski irrationality measure function, "second approximation" functions and multidimensional analogs. We discuss the attempts to find a generalization of the result by I. Kan and N. Moshchevitin about oscillation of the difference

$$\psi_\alpha(t) - \psi_\beta(t).$$

This research is supported by the Russian Science Foundation (project 18-41-05001).

HIGHEST ABUNDANT NUMBERS AND THE RIEMANN HYPOTHESIS

Oleg R. Musin (University of Texas Rio Grande Valley, Brownsville, USA)

In this paper we define Highest Abundant (HA) numbers [1]. Properties of these numbers are very different depending on whether the Riemann Hypothesis (RH) is true or false.

Let $n \geq 2$ and s be a real number. Define

$$R_s(n) := (e^{\gamma} n \log\log n - \sigma(n))\,(\log n)^s,$$

where $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function and γ is Euler's constant. Robin [3] showed that the RH is true if and only if

$$R_0(n) > 0 \ \text{ for all } \ n > 5040.$$

We say that n is HA with respect to R_s and write $n \in HA_s$ if for some real a, $R_s(k) - ak$ attains its minimum on $D = \{k \in \mathbb{N} \mid k \geq 5040\}$ at n.

In 1915, Ramanujan proved asymptotic inequalities for $\sigma(n)$ of the colossally abundant numbers (CA), assuming the RH [2]. Using a strong version of Ramanujan's theorem and Robin's inequalities we prove the following

Theorem. *(i) If the RH is true and $s > 1/2$, then there are infinitely many HA numbers with respect to R_s. If the RH is false, then HA_s is empty.*

(ii) Let $s \leq 0$. If the RH is false, then there are infinitely many HA numbers with respect to R_s. If the RH is true, then $HA_s = \{5040\}$.

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DIOPHANTINE APPROXIMATION IN CANTOR SETS

Johannes Schleischitz (Middle East Technical University, Northern Cyprus Campus, Turkey)

In 1984, K. Mahler asked how well irrational numbers in his Cantor middle-third set can be approximated by a) rational numbers inside the Cantor middle-third set b) rational numbers outside of it. Only in the last couple of years this particular question has been addressed in a few papers, in particular two papers by Fishman and Simmons, in context of more general fractal sets (including all "missing digit Cantor sets"). After an exposition of the mentioned results, we generalize and improve some of them. We further answer a question raised by Fishman and Simmons, showing that any missing digit Cantor set contains an irrational numbers with almost all its convergents contained in it (joint work with D. Roy). We also address the question of the algebraic structure of rational numbers in missing digit Cantor sets, thereby we obtain an improvement of result of Bloshchitsyn from 2015, and discuss several other related topics.

INTERSECTIONS OF BINARY QUADRATIC FORMS IN PRIMES AND THE PAUCITY PHENOMENON

Alisa Sedunova (MPIM Bonn, Bonn, Germany)

Let $r(n)$ be the function that counts the number of ways to represent a natural number n as a sum of two positive squares. The number of solutions to $a^2 + b^2 = c^2 + d^2 \leq x$, hence, the second moment of $r(n)$, in integers is well-known, while if one restricts all the variables to primes Erdős showed that only the diagonal solutions, namely, the ones with $a = c$, $b = d$ contribute to the main term, hence there is a paucity of the off-diagonal solutions. In this talk we discuss the second moments of $r(n)$, when *some* of the four variables are restricted to primes. In particular, we study the paucity phenomenon of the off-diagonal solutions in such problems. These mean values are not in a scope of the circle method, since the major arcs are too small due to the primality of certain variables, hence we shall develop other tools. The methods are largely based on Hooley's technique to tackle (on GRH) the Hardy-Littlewood problem about the representations $N = p + a^2 + b^2$, where p is a prime and a, b are integers, some related works of Plaksin (based on the unconditional resolution of the Hardy-Littlewood problem by Linnik), twists of Hooley delta function (which caused the final resolution of the Manin conjecture for Châtelet surfaces) and, lastly, more recent results of S. Daniel.

IRRATIONALITY EXPONENTS OF CERTAIN ALTERNATING SERIES

Iekata Shiokawa (Keio University, Yokohama, Japan)

Davison and Shallit defined the sequence $\{q_n\}$ by $q_0 = 1$, $q_1 = w_0$, $q_{n+1} = q_{n-1}(w_n q_n + 1)\,(n \geq 0)$, where $\{w_n\}$ is any sequence of positive integers. They give the continued fraction of the number $\sum_{n=0}^{\infty}(-1)^n/q_n q_{n+1}$ and proved its transcendence. In particular, transcendence of Cahen's constant $C = \sum_{n=0}^{\infty}(-1)^n/(S_n - 1)$ was established, where $S_0 = 2$, $S_{n+1} = S_n^2 - S_n + 1\,(n \geq 0)$ is Sylvester's sequence. Recently, the author jointly with Duverney and Kurosawa proved that for a positive integer l and algebraic numbers $a \neq 0$ and ρ with $S_n \neq \rho$ for all $n \geq 0$, the number $\sum_{n=0}^{\infty}(-1)^n/(S_n - \rho)^l$ is transcendental except when $l = a = 1$ and $\rho = 0$.

For sequences $\{w_n\}$ and $\{y_n\}$ of positive integers, we define the sequence $\{q_n\}$ by

$$q_0 = 1,\ q_1 = w_0,\ q_{n+1} = q_{n-1}(w_n q_n + y_n) \qquad (n \geq 1).$$

We consider the series

$$\xi = \sum_{n=0}^{\infty}(-1)^n \frac{y_1 y_2 \cdots y_{n+1}}{q_n q_{n+1}}$$

We give exact values of the irrationality exponents $\mu(\xi)$ of the number ξ assuming that $\log|y_n| = o(\alpha^n)$, where α is the golden ratio $(1+\sqrt{5})/2$. For example, we have

$$\mu\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{q_n q_{n+1}}\right) = 1 + \alpha$$

for any $\{w_n\}$ satisfying $\sum_{k=0}^{\infty} \log w_n/\alpha^n < \infty$, and also

$$\mu\left(\sum_{n=0}^{\infty}(-1)^n \frac{y_0 y_1 \cdots y_n}{S_n - 1}\right) = 3.$$

As we have $\mu(\xi) \geq 1 + \alpha$, all the numbers ξ are transcendental.

ACTIONS SL_2 AND MODULAR HYPERBOLAS

Ilya Shkredov (Steklov Mathematical Institute, Moscow, Russia)

We will talk about the connection of growth in SL_2 with a series of problems of Number Theory and Additive Combinatorics, namely, with Zaremba's conjecture on continued fractions, distribution of points on modular hyperbolas and bounds for bilinear Kloosterman sums.

THE SUCCESSIVE MINIMA FUNCTIONS ASSOCIATED TO SIMULTANEOUS APPROXIMATION OF m LINEARLY DEPENDENT REALS

Leonhard Summerer (Faculty of Mathematics, University of Vienna, Vienna, Austria)

In order to analyse the simultaneous approximation properties of m real numbers $\xi_1, \ldots, \xi_m$, the parametric Geometry of Numbers studies the joint behaviour of the successive minima functions with respect to a family of convex bodies $\mathcal{B}(q)$ varying with a parameter q and a lattice $\Lambda(\xi)$ defined in terms of $\xi_1, \ldots, \xi_m$. For simultaneuous approximation in the sense of Dirichlet, the linear independence of $1, \xi_1, \ldots, \xi_m$ implies a certain nice intersection property that any two consecutive minima functions enjoy:

Proposition 1. *Suppose* $1, \xi_1, \ldots, \xi_m$ *are linearly independent over* $\mathbb{Q}$ *and let* $\lambda_i(q)$ *denote the successive minima with respect to* $\Lambda(\xi)$ *and* $\mathcal{B}(q)$*. Then for every* $s \leq m$ *there exist arbitrarily large values of* q *for which* $\lambda_s(q) = \lambda_{s+1}(q)$*.*

An analoguous result holds in the more general situation where a system of exponents $(1, -\nu_1, \ldots, -\nu_m)$ with $\nu_i > 0$ for $1 \leq i \leq m$ and $\nu_1 + \ldots + \nu_m = 1$ is considered, which generalizes Dirichlet's classical simultaneous approximation in a way to consider nontrivial solutions of the system

$$\begin{aligned} |x| &\leq e^q \\ |\xi_1 x - y_1| &\leq e^{-\nu_1 q} \\ &\vdots \\ |\xi_m x - y_m| &\leq e^{-\nu_m q}. \end{aligned}$$

In this talk I will discuss whether the condition of linear independence of $1, \xi_1, \ldots, \xi_m$ in Proposition 1 is also necessary to guarantee that for given s we have arbitrarily large values of q with $\lambda_s(q) = \lambda_{s+1}(q)$. It will turn out that this is the case in the classical case where $\nu_i = 1/m$, however, for arbitrary systems of exponents this is no longer the case.

For a general system of exponents I will state a criterion that allows to decide if $L_s(q) < L_{s+1}(q)$ for all large q respectively if $L_s(q) = L_{s+1}(q)$ holds for some arbitrarily large q with $1 \leq s \leq m$ when a dependence relation between $1, \xi_1, \ldots, \xi_m$ is given.

CERTAIN EXPONENTIAL DIOPHANTINE EQUATIONS

László Szalay (Institute of Mathematics, University of Sopron, Sopron, Hungary)

A remarkable class of diophantine equations is called exponential. In the talk, we recall a few types of

$$a_1 b_{11}^{x_{11}} \cdots b_{1s}^{x_{1s}} + \cdots + a_r b_{r1}^{x_{r1}} \cdots b_{rs}^{x_{rs}} = c,$$

where the non-negative integer unknowns are the exponents x_{ij}. Furthermore we study some recent results, for example the explicit solutions to

$$(2^k - 1)(3^\ell - 1) = (5^m - 1).$$

HEIGHT PAIRINGS ON SPLIT TORI

Valerio Talamanca (Università Roma Tre, Roma, Italia)

Let $\mathbb{G}_m$ denote the d-dimensional split torus defined over a number field k. By means of the spectral height we associate to each $\mathbb{G}_m$-module E a height function h_E on $\mathbf{GL}(E)$. This gives rise to a height pairing between the monoid of irreducible $\mathbb{G}_m$-modules of $\mathbb{G}_m$ and the group $\mathbb{G}_m\left(\overline{k}\right)$. Our main results are the follwing: a characterization of those $\mathbb{G}_m$-modules E for which h_E satisfeis Northcott's finiteness theorem, the determination of the kernels of the height pairing, as well as, for a few special classes of $\mathbb{G}_m$-modules, of the group of automorphisms that preserve h_E.

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EQUIDISTRIBUTION, VAN DER CORPUT SETS AND EXPONENTIAL SUMS

Robert Tichy (Technische Universität Graz, Graz, Austria)

A set $H \subseteq \mathbb{Z}$ is called a van der Corput set if any sequence $(x_n)_{n\in\mathbb{N}}$ is equidistributed provided that the difference sequences $(x_{n+h}-x_n)_{n\in\mathbb{N}}$ are equidistributed for all $h \in H$. By van der Corput's difference theorem, $H = \mathbb{N}$ is a van der Corput set. This concept is related to sets of recurrence and difference sets and other concepts from additive combinatorics. We establish new results on sets of recurrence and van der Corput sets in $\mathbb{Z}^k$ (i.e. for $\mathbb{Z}^k$-actions) which refine and unify some of the previous results obtained by Sarközy, Furstenberg, Kamae and M'endes France, and Bergelson and Lesigne. Furthermore, we construct some new examples of such sets involving prime numbers. This involves new bounds for exponential sums containing generalized polynomials of the form

$$f(x) = \sum_{j=1}^{m} \alpha_j x^{\theta_j},$$

where $0 < \theta_1 < \theta_2 < \cdots < \theta_m, \alpha_j$ are non-zero reals and at least one α_j is irrational if all $\theta_j \in \mathbb{N}$. Furthermore, we apply this method to diophantine inequalities involving prime numbers p. As a special result we obtain

$$\min_{1\le p\le N} \|\xi \lfloor f(p) \rfloor \| \ll N^{-\eta},$$

where ξ is a real number, N a sufficiently large positive integer and $\|\cdot\|$ denotes the distance to the nearest integer, $\lfloor f(p) \rfloor$ the floor function and $\eta > 0$ a suitable exponent. This is recent joint work with Manfred Madritsch.

NORMAL NUMBERS IN FRACTALS

Barak Weiss (Tel-Aviv University, Israel)

We classify stationary measures for some random walks driven by a semigroup of affine maps of a torus. As a corollary, we exhibit fractals with the property that a.e. point on the fractal (w.r.t. the natural Hausdorff measure or any Bernoulli measure) is generic in base D. For example, if $\mathcal{C}$ is the dilation of Cantor's middle thirds set by an irrational number, then a.e. point on $\mathcal{C}$ is generic for base 3.

Joint work with Yiftach Dayan and Arijit Ganguly.

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