Let \( \{A_n\} \) be a nonzero sequence of complex numbers. In [5, 6, 13], it was proved that if
\[
A_{n+2}A_{n-2} = \alpha A_{n+1}A_{n-1} + \beta A_n^4,
\]
for some fixed \( \alpha \neq 0 \) and \( \beta \), then there exist complex numbers \( z, g_1, g_2, \) and \( g_3 \) such that
\[
A_n = C D^n \frac{\sigma(z_0 + nz)}{\sigma(z)^n},
\]
where
\[
C = \frac{A_0}{\sigma(z_0)}, \quad D = \frac{\sigma(z_0) A_1}{\sigma(z + z_0) A_0}
\]
and \( \sigma \) is the Weierstrass function associated with the elliptic curve
\[
y^2 = 4x^3 - g_2x - g_3.
\]
Moreover, if curve (3) is singular, i.e., its discriminant \( g_3^2 - 27g_2^2 \) vanishes, then the Weierstrass \( \sigma \)-function can be replaced by its degenerate analogues (see [1])
\[
\sigma(z) = z, \quad (g_2 = g_3 = 0, \omega_1 = \omega_2 = \infty),
\]
\[
\sigma(z) = \frac{2\omega}{\pi} \exp \left( i \frac{\pi z}{2\omega} \right) \sin \frac{\pi z}{2\omega}
\]
\[
(g_2, g_3 \neq 0, \omega_1 = \omega, \omega_2 = \infty).
\]

The Somos-4 sequences are closely related to elliptic divisibility sequences (see [12, 13, 15]) and integrable discrete-time dynamical systems (see [3, 6, 8, 9, 11, 14]). One of the fundamental properties of a Somos-4 sequence is the identity (see [10])
\[
\tau_{m_i+n_i} \tau_{m_i-n_i} = \tau_{m_i+n_i} \tau_{m_i-n_i} \tau_{m_i+n_i} \tau_{m_i-n_i} = 0, \quad (4)
\]
which holds for any integer or simultaneously half-integer \( m_i, n_i \) \( (i = 1, 2, 3) \). In particular, for \( m_1 = n, m_2 = 1, m_3 = 0, n_1 = 2, n_2 = 1, \) and \( n_3 = 0, \) this identity transforms into the initial recursive relation (1). Verifying that (4) holds for sequence (2) reduces to applying the Weierstrass three-term identity (see [2])
\[
\sigma(a + b)\sigma(a - b)\sigma(c + d)\sigma(c - d) -
- \sigma(a + c)\sigma(a - c)\sigma(b + d)\sigma(b - d) +
+ \sigma(a + d)\sigma(a - d)\sigma(b + c)\sigma(b - c) = 0.
\]
In the same way, we verify that the sequences
\[
A_n = C_1 D_1^n \frac{\sigma(z_1 + nz)}{\sigma(z)^n}, \quad B_n = C_2 D_2^n \frac{\sigma(z_2 + nz)}{\sigma(z)^n}
\]
(5)
satisfy the following relation similar to (4):
\[
\left| \begin{array}{cccc}
A_{m_1+n_1}B_{m_1-n_1} & A_{m_2+n_2}B_{m_2-n_2} & A_{m_3+n_3}B_{m_3-n_3} \\
A_{m_1+n_1}B_{m_1-n_1} & A_{m_2+n_2}B_{m_2-n_2} & A_{m_3+n_3}B_{m_3-n_3} \\
A_{m_1+n_1}B_{m_1-n_1} & A_{m_2+n_2}B_{m_2-n_2} & A_{m_3+n_3}B_{m_3-n_3}
\end{array} \right| = 0. \quad (6)
\]

Consider the more general problem of finding sequences \( \{A_n\}, \{B_n\} \subset C \) specified by their initial terms
\[
A_{k_1}, A_{k_2}, A_0, B_{k_2}, B_{k_1}, B_0
\]
(7)
and the recurrence relations
\[
A_{n+2}B_{n+2} - \alpha A_{n+1}B_{n+1} + \beta A_nB_n, \quad (8)
\]
\[
A_{n-2}B_{n+2} = \gamma A_{n-1}B_{n+1} + \delta A_nB_n. \quad (9)
\]

We assume that the sequences contain no zero elements, because otherwise, relations (8) and (9) do not determine them for all integer \( n \). A general solution of this problem is not described by (5), because the sequences \( \{A_n\}, \{B_n\} \) are determined by 12 free parameters (the 10 initial conditions (7) and the four coeffi-

cients \( \alpha, \beta, \gamma, \delta \) related by the two linear equations obtained from (8) and (9) at \( n = 0 \), while expressions (5) use only nine free parameters \( (C_{1,2}, D_{1,2}, z, \sigma_{1,2}, \text{and } g_{1,2}) \). However, if the initial terms of the sequences \( \{A_n\}, \{B_n\} \) are assumed to satisfy (6), then the general solution indeed has the form (5).

We assume that the initial conditions (7) are related by
\[
\begin{pmatrix} A_2B_0 & A_1B_1 & A_0B_2 \\ A_1B_1 & A_0B_0 & A_1B_1 \\ A_0B_1 & A_1B_0 & A_2B_0 \end{pmatrix} = 0, \tag{10}
\]
and \( A_1 \) and \( B_1 \) obtained from (8) and (9) at \( n = 1 \) satisfy the relations
\[
\begin{pmatrix} A_1B_0 & A_1B_1 & A_2B_1 \\ A_2B_1 & A_1B_0 & A_0B_1 \\ A_2B_1 & A_1B_0 & A_1B_0 \end{pmatrix} = 0, \tag{11}
\]
\[
\begin{pmatrix} A_1B_1 & A_2B_1 & A_3B_1 \\ A_2B_2 & A_1B_1 & A_2B_1 \\ A_1B_1 & A_2B_1 & A_3B_1 \end{pmatrix} = 0. \tag{12}
\]

This means, in particular, that the coefficients \( \alpha, \beta, \gamma, \delta \) of the recurrence relations (8) and (9) are uniquely determined by the initial terms (7). Indeed, \( A_3 \) and \( B_3 \) can be found from relations (11) and (12). Relation (8) with \( n = 0 \) and \( n = 1 \) gives the following system of two linear equations, which uniquely determines the unknowns \( \alpha \) and \( \beta \):
\[
\begin{pmatrix} A_1B_{n-1} & A_1B_1 \\ A_2B_{n-2} & A_2B_0 \end{pmatrix} = \frac{A_2B_0}{\Delta_1}, \quad \beta = \frac{A_2B_0}{\Delta_1}, \tag{13}
\]
\[
\Delta_1 = \begin{pmatrix} A_2B_0 & A_1B_1 \\ A_2B_1 & A_1B_0 \end{pmatrix}. \]

The parameters \( \gamma \) and \( \delta \) are found in a similar way:
\[
\begin{pmatrix} A_{-1}B_3 & A_3B_1 \\ A_2B_2 & A_0B_0 \end{pmatrix} = \frac{A_0B_0}{\Delta_2}, \quad \delta = \frac{A_0B_0}{\Delta_2}, \tag{14}
\]
\[
\Delta_2 = \begin{pmatrix} A_0B_0 & A_1B_1 \\ A_1B_1 & A_2B_0 \end{pmatrix}.
\]

Thus, the problem stated above can be reformulated as follows: describe complex sequences \( \{A_n\}, \{B_n\} \) determined by the initial terms (7) whose terms \( A_3 \) and \( B_3 \) are determined from (11) and (12) and the remaining terms are calculated by the recurrence relations

\[
\begin{align*}
A_{n+2}B_{n-2} - A_{n+1}B_{n-1} - A_nB_n &= 0, \tag{15} \\
A_{n+2}B_1 &- A_{n+1}B_0 = 0, \tag{16}
\end{align*}
\]

To ensure that \( A_3, B_3, \alpha, \beta, \gamma, \delta \) are well-defined, we must require that \( \Delta_1 \Delta_2 \neq 0 \), where \( \Delta_1, \Delta_2 \) are determined by (13) and (14), respectively, and
\[
\Delta_0 = \begin{pmatrix} A_0B_0 \end{pmatrix}.
\]

Remark. Relations (10)–(12), (15), and (16) are obtained from (6) at
\[
\begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3/2 & 1/2 & -1/2 \end{pmatrix},
\]
\[
\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} n & 0 \end{pmatrix}, \quad \begin{pmatrix} n & 1 \end{pmatrix}, \quad \begin{pmatrix} -2 & -1 \end{pmatrix},
\]
respectively.

Theorem. A general solution of the problem stated above has the form (5), where
\[
C_1 = \frac{A_0}{\sigma(z_1)}, \quad C_2 = \frac{B_0}{\sigma(z_2)}, \quad D_i = \frac{\sigma(z)\sigma(z_i)A_i}{\sigma(z + z_i)A_0},
\]
where \( \sigma \) is the Weierstrass function (possibly degenerate) associated with a curve of the form (3).

If the sequences \( \{A_n\} \) and \( \{B_n\} \) satisfy relations (8) and (9), then, obviously, the sequences
\[
\tilde{A}_n = \tilde{C}_1 \tilde{D}_1^n A_0, \quad \tilde{B}_n = \tilde{C}_2 \tilde{D}_2^n A_0
\]
satisfy (8) and (9) as well for any \( \tilde{C}_1, \tilde{D}_1, \tilde{C}_2, \tilde{D}_2 \). Therefore, it is natural to pass to the gauge-invariant variables
\[
a_n = \frac{A_{n+1}A_{n-1}}{A_n^2}, \quad b_n = \frac{B_{n+1}B_{n-1}}{B_n^2}, \tag{17}
\]
for which condition (10) can be rewritten in the form
\[
\begin{pmatrix} a_n/b_0^n & 1/b_n/a_0 \\ 1 & 1 \end{pmatrix} = 0, \tag{18}
\]
or in the equivalent form
\[
a_n a_{-1}^2 a_n - b_{n+1} b_n b_1 = a_n b_0 (a_{-1} a_1 - b_{n+1} (b_{-1} + b_1)). \tag{19}
\]
Proposition. If parameters \( a_0, a_{\pm 1}, b_0, b_{\pm 1} \in C^* \) are related by (18) and \( a_0 \neq b_0 \), then there exists a curve
\[
y^2 = 4x^3 - g_2x - g_3
\]
and numbers \( z_1, z_2, z \in C \) such that
\[
a_n = \varphi(z) - \varphi(x_i + nz),
b_n = \varphi(z) - \varphi(x_i + nz) \quad (n = 0, \pm 1).
\]

Proof. We follow the same line of reasoning as in the proof of Proposition 2.2 in [6]. We successively determine the coordinates of the points
\[
(\lambda, \mu) = (\varphi(z), \varphi'(z)), \quad (v_i, \xi_i) = (\varphi(z_i), \varphi'(z_i)) \quad (i = 1, 2),
\]
on curve (20). Obviously, we have
\[
a_0 = \lambda - v_1, \quad b_0 = \lambda - v_2.
\]
It follows from the identity (see [5])
\[
(\varphi(z) - \varphi(u - z))(\varphi(z) - \varphi(u))(\varphi(z) - \varphi(u + z)) = \varphi'(z)^2 - \varphi'(z)^2 \left( \varphi(2z) - \varphi(z) \right) / \varphi(z) - \varphi(u)
\]
that
\[
a_{-1}a_0^2a_1 = a_0\mu^2 + (\varphi(2z) - \varphi(z))\mu^2,
\]
\[
b_{-1}b_0^2b_1 = b_0\mu^2 + (\varphi(2z) - \varphi(z))\mu^2.
\]
Therefore, \( \mu \) is determined by
\[
\mu^2 = \frac{a_{-1}a_0^2a_1 - b_{-1}b_0^2b_1}{a_0 - b_0}.
\]
According to the addition formula (see [1, 2])
\[
\varphi(u + v) = \frac{1}{4} \left( \frac{\varphi'(u) - \varphi'(v)}{\varphi(u) - \varphi(v)} \right)^2 - \varphi(u) - \varphi(v),
\]
we have
\[
a_{\pm 1} = \frac{1}{4} \left( \frac{\xi_1 + \mu}{a_0} \right)^2 + v_1 + 2\lambda,
b_{\pm 1} = \frac{1}{4} \left( \frac{\xi_2 + \mu}{b_0} \right)^2 + v_2 + 2\lambda.
\]
Considering the differences \( a_i = a_{-1} \) and \( b_i = b_{-1} \), we see that
\[
\xi_{\pm 1}\mu = \Delta_a, \quad \xi_{\pm 2}\mu = \Delta_b,
\]
where
\[
\Delta_a = a_0^2(a_{-1}), \quad \Delta_b = b_0^2(b_{-1}).
\]
First, suppose that \( \mu \neq 0 \). Then \( \xi_{1} \) and \( \xi_{2} \) can be found from (27). Writing relations (26) in the form
\[
v_1 + 2\lambda = a_1 + \frac{(\Delta_a - \mu^2)^2}{2a_0\mu},
v_2 + 2\lambda = b_1 + \frac{(\Delta_b - \mu^2)^2}{2b_0\mu},
\]
and combining them with (23), we obtain a system of linear equations, which determines \( v_1, v_2, \) and \( \lambda \) (condition (18) ensures the consistency of this system). The parameters \( g_2 \) and \( g_3 \) are found from the system
\[
\mu^2 = 4\lambda^3 - g_2\lambda - g_3,
\]
\[
\xi_1^2 = 4v_1^3 - g_2v_1 - g_3 \quad (i = 1, 2),
\]
whose consistence is ensured by the same condition (18). Thus, the coordinates of points (22) are determined uniquely up to the involution \( (\lambda, \mu) \to (\lambda, -\mu), (v_i, \xi_i) \to (v_i, -\xi_i) \), which corresponds to the change \( z \to -z, \ z_i \to -z_i \ (i = 1, 2) \). A direct verification shows that the value \( \varphi(2z) - \varphi(z) \) determined by the doubling formula (see [1, 2])
\[
\varphi(2z) = \frac{1}{4} \left( \frac{\varphi'(z)^2}{\varphi(z)} \right)^2 - 2\varphi(z) = \left( \frac{12\lambda^2 - g_2}{4\mu} \right)^2 - 2\lambda
\]
(under condition (18)) satisfies Eqs. (24) and (25). If \( g_2^3 = 27g_3^2 \), then curve (20) is singular, and the Weierstrass elliptic function in (21) should be replaced by its degenerate analogues (see [1])
\[
\varphi(z) = \frac{1}{\xi^2} \quad (g_2 = g_3 = 0, \omega_1 = \omega_2 = \infty),
\]
\[
\varphi(z) = \left( \frac{\pi}{2\omega_0} \right)^2 \left( -\frac{1}{3} + \sin \frac{\pi z}{2\omega_0} \right)^2.
\]
Now, consider the case where \( \mu = 0 \). In this case, we have \( a_{-1}a_0^2a_1 = b_{-1}b_0^2b_1 \), and condition (18) implies \( a_0(a_{-1} + a_1) = b_0(b_{-1} + b_1) \). It follows from (27) that \( a_{-1} = b_{-1} \), and \( b_{-1} = b_{-1} \), i.e. \( a_0a_1 = b_0b_1 \).

Let us successively express the unknowns in terms of the parameter \( \lambda \). From (23) and (26) we find
\[
v_1 = \lambda - a_0, \quad v_2 = \lambda - b_0,
\]
\[
\xi_1^2 = 4a_0^2(3\lambda - a_0 - a_1), \quad \xi_2^2 = 4b_0^2(3\lambda - b_0 - b_1).
\]
Substituting \( \xi_1^2 \) and \( \xi_2^2 \) into (30), we obtain the relations
\[
12\lambda^2 - g_2 + g_3/a_0 - 4a_0a_1 = 0,
12\lambda^2 - g_2 + g_3/b_0 - 4b_0b_1 = 0,
\]
which imply \( g_3 = 0 \) and \( 12\lambda^2 - g_2 - 4a_0a_1 = 0 \). Moreover, according to (29), we have \( 4\lambda^3 - g_2\lambda = 0 \). There-
fore, λ = 0 or λ = ±\sqrt{\alpha_0/2}. In each of these cases, as well as at μ ≠ 0, parameters (22) are determined uniquely up to the involution (λ, μ) → (λ, −μ), (v_i, ξ_i) → (v_i, −ξ_i) (i = 1, 2).

Proof of the theorem. The proposition proved above and the relation (see [1, 2]) imply (5) for −2 ≤ n ≤ 2. Since the remaining terms of the sequences \{A_n\} and \{B_n\} are determined by relations (11)–(16) and the sequences \(C_iD_i^n \frac{\sigma(z_i + nz)}{\sigma(z)^n}\), \(i = 1, 2\) satisfy the same relations (see the remark to the theorem), it follows that (5) holds for all integer n.

Corollary. Special cases of relation (6) are

\[
\begin{align*}
A_{2n}B_0 & \quad A_{n+1}B_{n-1} \quad A_nB_n \\
A_{1+n}B_{1-n} & \quad A_1B_0 \quad A_1B_1 = 0, \\
A_{n-1}B_n & \quad A_0B_{n-1} \quad A_0B_0 \\
A_{2n-1}B_0 & \quad A_nB_{n-1} \quad A_{n-1}B_n \\
A_{n-1}B_n & \quad A_0B_{n-1} \quad A_0B_0 \\
A_{1-n}B_{1+n} & \quad A_0B_2 \quad A_1B_1 = 0, \\
A_{n-1}B_n & \quad A_1B_1 \quad A_0B_0 \\
A_{2n-1}B_n & \quad A_{n-1}B_{n-1} \quad A_{n-1}B_{n-1} = 0, \\
A_{1-n}B_n & \quad A_0B_1 \quad A_0B_0 \quad A_0B_1 = 0, \\
A_{n-1}B_{n-1} & \quad A_0B_1 \quad A_0B_0 \quad A_0B_1 = 0,
\end{align*}
\]

which can also be used to calculate the elements of the sequences \(\{A_n\}\) and \(\{B_n\}\). It follows that all terms of these sequences are Laurent polynomials in the initial data and the parameters \(\Delta_0, \Delta_1, \Delta_2\).

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\[
A_n, B_n \in \mathbb{Z}^\left[ A_0^{\pm 1}, B_0^{\pm 1}, A_{\pm 1}, A_{\pm 2}, B_{\pm 1}, B_{\pm 2}, \Delta_0^{\pm 1}, \Delta_1^{\pm 1}, \Delta_2^{\pm 1}\right].
\]

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