ISSN 1064-5624, Doklady Mathematics, 2016, Vol. 94, No. 3, pp. 1–4. © Pleiades Publishing, Ltd., 2016.
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Somos-4 and Elliptic Systems of Sequences

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Received June 30, 2016

Abstract—A general formula for elements of double Somos-4 sequences is obtained. A sufficient integrality condition for such sequences is presented.

DOI: 10.1134/S1064562416060016

Let $\{A_n\}$ be a nonzero sequence of complex numbers. In [5, 6, 13], it was proved that if

$$A_{n+2}A_{n-2} = \alpha A_{n+1}A_{n-1} + \beta A_n^2, \tag{1}$$

for some fixed $\alpha \neq 0$ and β, then there exist complex numbers $z \neq 0$, z_0, g_2, and g_3 such that

$$A_n = CD^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}}, \tag{2}$$

where

$$C = \frac{A_0}{\sigma(z_0)}, \quad D = \frac{\sigma(z)\sigma(z_0)A_1}{\sigma(z + z_0)A_0}$$

and σ is the Weierstrass function associated with the elliptic curve

$$y^2 = 4x^3 - g_2x - g_3. \tag{3}$$

Moreover, if curve (3) is singular, i.e., its discriminant $g_2^3 - 27g_3^2$ vanishes, then the Weierstrass σ-function can be replaced by its degenerate analogues (see [1])

$$\sigma(z) = z \quad (g_2 = g_3 = 0, \omega_1 = \omega_2 = \infty),$$

$$\sigma(z) = \frac{2\omega}{\pi}\exp\left(\frac{1}{3!}\left(\frac{\pi z}{2\omega}\right)^2\right)\sin\frac{\pi z}{2\omega}$$
$$(g_2, g_3 \neq 0, \omega_1 = \omega, \omega_2 = \infty).$$

The Somos-4 sequences are closely related to elliptic divisibility sequences (see [12, 13, 15]) and integrable discrete-time dynamical systems (see [3, 6, 8, 9, 11, 14]). One of the fundamental properties of a Somos-4 sequence is the identity (see [10])

$$\begin{vmatrix} \tau_{m_1+n_1}\tau_{m_1-n_1} & \tau_{m_1+n_2}\tau_{m_1-n_2} & \tau_{m_1+n_3}\tau_{m_1-n_3} \\ \tau_{m_2+n_1}\tau_{m_2-n_1} & \tau_{m_2+n_2}\tau_{m_2-n_2} & \tau_{m_2+n_3}\tau_{m_2-n_3} \\ \tau_{m_3+n_1}\tau_{m_3-n_1} & \tau_{m_3+n_2}\tau_{m_3-n_2} & \tau_{m_3+n_3}\tau_{m_3-n_3} \end{vmatrix} = 0, \tag{4}$$

which holds for any integer or simultaneously half-integer m_i, n_i $(i = 1, 2, 3)$. In particular, for $m_1 = n$, $m_2 = 1$, $m_3 = 0$, $n_1 = 2$, $n_2 = 1$, and $n_3 = 0$, this identity transforms into the initial recursive relation (1). Verifying that (4) holds for sequence (2) reduces to applying the Weierstrass three-term identity (see [2])

$$\sigma(a + b)\sigma(a - b)\sigma(c + d)\sigma(c - d) -$$
$$- \sigma(a + c)\sigma(a - c)\sigma(b + d)\sigma(b - d) +$$
$$+ \sigma(a + d)\sigma(a - d)\sigma(b + c)\sigma(b - c) = 0.$$

In the same way, we verify that the sequences

$$A_n = C_1D_1^n \frac{\sigma(z_1 + nz)}{\sigma(z)^{n^2}}, \quad B_n = C_2D_2^n \frac{\sigma(z_2 + nz)}{\sigma(z)^{n^2}} \tag{5}$$

satisfy the following relation similar to (4):

$$\begin{vmatrix} A_{m_1+n_1}B_{m_1-n_1} & A_{m_1+n_2}B_{m_1-n_2} & A_{m_1+n_3}B_{m_1-n_3} \\ A_{m_2+n_1}B_{m_2-n_1} & A_{m_2+n_2}B_{m_2-n_2} & A_{m_2+n_3}B_{m_2-n_3} \\ A_{m_3+n_1}B_{m_3-n_1} & A_{m_3+n_2}B_{m_3-n_2} & A_{m_3+n_3}B_{m_3-n_3} \end{vmatrix} = 0. \tag{6}$$

Consider the more general problem of finding sequences $\{A_n\}$, $\{B_n\} \subset C$ specified by their initial terms

$$A_{\pm 2}, A_{\pm 1}, A_0, B_{\pm 2}, B_{\pm 1}, B_0 \tag{7}$$

and the recurrence relations

$$A_{n+2}B_{n-2} = \alpha A_{n+1}B_{n-1} + \beta A_nB_n, \tag{8}$$

$$A_{n-2}B_{n+2} = \gamma A_{n-1}B_{n+1} + \delta A_nB_n. \tag{9}$$

We assume that the sequences contain no zero elements, because otherwise, relations (8) and (9) do not determine them for all integer n. A general solution of this problem is not described by (5), because the sequences $\{A_n\}$, $\{B_n\}$ are determined by 12 free parameters (the 10 initial conditions (7) and the four coeffi-

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cients $\alpha, \beta, \gamma, \delta$related by the two linear equations obtained from (8) and (9) at $n = 0$), while expressions (5) use only nine free parameters $(C_{1,2}, D_{1,2}, z, z_{1,2}$, and $g_{2,3})$. However, if the initial terms of the sequences$\{A_n\}$, $\{B_n\}$ are assumed to satisfy (6), then the general solution indeed has the form (5).

We assume that the initial conditions (7) are related by

$$\begin{vmatrix} A_2B_0 & A_1B_1 & A_0B_2 \\ A_1B_{-1} & A_0B_0 & A_{-1}B_1 \\ A_0B_{-2} & A_{-1}B_{-1} & A_{-2}B_0 \end{vmatrix} = 0, \tag{10}$$

and A_3 and B_3 obtained from (8) and (9) at $n = 1$ satisfy the relations

$$\begin{vmatrix} A_3B_0 & A_2B_1 & A_1B_2 \\ A_2B_{-1} & A_1B_0 & A_0B_1 \\ A_1B_{-2} & A_0B_{-1} & A_{-1}B_0 \end{vmatrix} = 0, \tag{11}$$

$$\begin{vmatrix} A_2B_1 & A_1B_2 & A_0B_3 \\ A_1B_0 & A_0B_1 & A_{-1}B_2 \\ A_0B_{-1} & A_{-1}B_0 & A_{-2}B_1 \end{vmatrix} = 0. \tag{12}$$

This means, in particular, that the coefficients $\alpha, \beta, \gamma, \delta$of the recurrence relations (8) and (9) are uniquely determined by the initial terms (7). Indeed, A_3 and B_3 can be found from relations (11) and (12). Relation (8) with $n = 0$ and $n = 1$ gives the following system of two linear equations, which uniquely determines the unknowns α and β:

$$\alpha = \frac{\begin{vmatrix} A_3B_{-1} & A_1B_1 \\ A_2B_{-2} & A_0B_0 \end{vmatrix}}{\Delta_1}, \quad \beta = \frac{\begin{vmatrix} A_2B_0 & A_3B_{-1} \\ A_1B_{-1} & A_2B_{-2} \end{vmatrix}}{\Delta_1}, \quad \Delta_1 = \begin{vmatrix} A_2B_0 & A_1B_1 \\ A_1B_{-1} & A_0B_0 \end{vmatrix}. \tag{13}$$

The parameters γ and δ are found in a similar way:

$$\gamma = \frac{\begin{vmatrix} A_{-1}B_3 & A_1B_1 \\ A_{-2}B_2 & A_0B_0 \end{vmatrix}}{\Delta_2}, \quad \delta = \frac{\begin{vmatrix} A_0B_2 & A_{-1}B_3 \\ A_{-1}B_1 & A_{-2}B_2 \end{vmatrix}}{\Delta_2}, \quad \Delta_2 = \begin{vmatrix} A_0B_2 & A_1B_1 \\ A_{-1}B_1 & A_0B_0 \end{vmatrix}. \tag{14}$$

Thus, the problem stated above can be reformulated as follows: describe complex sequences $\{A_n\}$, $\{B_n\}$ determined by the initial terms (7) whose terms A_3 and B_3 are determined from (11) and (12) and the remaining terms are calculated by the recurrence relations

$$\begin{vmatrix} A_{n+2}B_{n-2} & A_{n+1}B_{n-1} & A_nB_n \\ A_3B_{-1} & A_2B_0 & A_1B_1 \\ A_2B_{-2} & A_1B_{-1} & A_0B_0 \end{vmatrix} = 0, \tag{15}$$

$$\begin{vmatrix} A_{n-2}B_{n+2} & A_{n-1}B_{n+1} & A_nB_n \\ A_{-1}B_3 & A_0B_2 & A_1B_1 \\ A_{-2}B_2 & A_{-1}B_1 & A_0B_0 \end{vmatrix} = 0. \tag{16}$$

To ensure that A_3, B_3, $\alpha, \beta, \gamma, \delta$ are well-defined, we must require that $\Delta_0\Delta_1\Delta_2 \neq 0$, where Δ_1, Δ_2 are determined by (13) and (14), respectively, and $\Delta_0 = \begin{vmatrix} A_1B_0 & A_0B_1 \\ A_0B_{-1} & A_{-1}B_0 \end{vmatrix}$.

Remark. Relations (10)–(12), (15), and (16) are obtained from (6) at $\begin{pmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 3/2 & 1/2 & -1/2 \\ 3/2 & 1/2 & -1/2 \end{pmatrix}$,

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} n & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} n & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}.,$$

respectively.

Theorem. *A general solution of the problem stated above has the form* (5), *where*

$$C_1 = \frac{A_0}{\sigma(z_1)}, \quad C_2 = \frac{B_0}{\sigma(z_2)}, \quad D_1 = \frac{\sigma(z)\sigma(z_1)A_1}{\sigma(z+z_1)A_0},$$
$$C_1 = \frac{\sigma(z)\sigma(z_2)B_1}{\sigma(z+z_2)B_0},$$

z, z_1, z_2 *are complex numbers, and* σ *is the Weierstrass function* (*possibly degenerate*) *associated with a curve of the form* (3).

If the sequences $\{A_n\}$ and $\{B_n\}$ satisfy relations (8) and (9), then, obviously, the sequences

$$\tilde{A}_n = \tilde{C}_1\tilde{D}_1^nA_n, \quad \tilde{B}_n = \tilde{C}_2\tilde{D}_2^nA_n$$

satisfy (8) and (9) as well for any $\tilde{C}_1$, $\tilde{D}_1$, $\tilde{C}_2$, $\tilde{D}_2$. Therefore, it is natural to pass to the gauge-invariant variables

$$a_n = \frac{A_{n+1}A_{n-1}}{A_n^2}, \quad b_n = \frac{B_{n+1}B_{n-1}}{B_n^2}, \tag{17}$$

for which condition (10) can be rewritten in the form

$$\begin{vmatrix} a_1/b_0 & 1 & b_1/a_0 \\ 1 & 1 & 1 \\ b_{-1}/a_0 & 1 & a_{-1}/b_0 \end{vmatrix} = 0 \tag{18}$$

or in the equivalent form

$$a_{-1}a_0^2a_1 - b_{-1}b_0^2b_1 = a_0b_0(a_0(a_{-1}+a_1) - b_0(b_{-1}+b_1)). \tag{19}$$

Proposition. *If parameters* $a_0, a_{\pm1}, b_0, b_{\pm1} \in C^*$ *are related by* (18) *and* $a_0 \neq b_0$, *then there exists a curve*

$$y^2 = 4x^3 - g_2 x - g_3 \tag{20}$$

and numbers $z_1, z_2, z \in C$ *such that*

$$\begin{aligned} a_n &= \wp(z) - \wp(z_1 + nz), \\ b_n &= \wp(z) - \wp(z_2 + nz) \quad (n = 0, \pm1). \end{aligned} \tag{21}$$

Proof. We follow the same line of reasoning as in the proof of Proposition 2.2 in [6]. We successively determine the coordinates of the points

$$\begin{aligned} (\lambda, \mu) &= (\wp(z), \wp'(z)), (\nu_i, \xi_i) \\ &= (\wp(z_i), \wp'(z_i)) \quad (i = 1, 2), \end{aligned} \tag{22}$$

on curve (20). Obviously, we have

$$a_0 = \lambda - \nu_1, \quad b_0 = \lambda - \nu_2. \tag{23}$$

It follows from the identity (see [5])

$$\begin{aligned} &(\wp(z) - \wp(u - z))(\wp(z) - \wp(u))(\wp(z) - \wp(u + z)) \\ &= \wp'(z)^2 - \frac{\wp'(z)^2 (\wp(2z) - \wp(z))}{\wp(z) - \wp(u)} \end{aligned}$$

that

$$a_{-1}a_0^2 a_1 = a_0\mu^2 + (\wp(2z) - \wp(z))\mu^2, \tag{24}$$

$$b_{-1}b_0^2 b_1 = b_0\mu^2 + (\wp(2z) - \wp(z))\mu^2. \tag{25}$$

Therefore, μ is determined by

$$\mu^2 = \frac{a_{-1}a_0^2 a_1 - b_{-1}b_0^2 b_1}{a_0 - b_0}.$$

According to the addition formula (see [1, 2])

$$\wp(u + v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2 - \wp(u) - \wp(v),$$

we have

$$\begin{aligned} a_{\pm1} &= -\frac{1}{4}\left(\frac{\xi_1 \mp \mu}{a_0}\right)^2 + \nu_1 + 2\lambda, \\ b_{\pm1} &= -\frac{1}{4}\left(\frac{\xi_2 \mp \mu}{b_0}\right)^2 + \nu_2 + 2\lambda. \end{aligned} \tag{26}$$

Considering the differences $a_1 - a_{-1}$ and $b_1 - b_{-1}$, we see that

$$\xi_1\mu = \Delta_a, \quad \xi_2\mu = \Delta_b, \tag{27}$$

where

$$\Delta_a = a_0^2(a_1 - a_{-1}), \quad \Delta_b = b_0^2(b_1 - b_{-1}). \tag{28}$$

First, suppose that $\mu \neq 0$. Then ξ_1 and ξ_2 can be found from (27). Writing relations (26) in the form

$$\nu_1 + 2\lambda = a_1 + \left(\frac{\Delta_a - \mu^2}{2a_0\mu}\right)^2,$$

$$\nu_2 + 2\lambda = b_1 + \left(\frac{\Delta_b - \mu^2}{2b_0\mu}\right)^2,$$

and combining them with (23), we obtain a system of linear equations, which determines ν_1, ν_2, and λ (condition (18) ensures the consistency of this system). The parameters g_2 and g_3 are found from the system

$$\mu^2 = 4\lambda^3 - g_2\lambda - g_3, \tag{29}$$

$$\xi_i^2 = 4\nu_i^3 - g_2\nu_i - g_3 \quad (i = 1, 2), \tag{30}$$

whose consistency is ensured by the same condition (18). Thus, the coordinates of points (22) are determined uniquely up to the involution $(\lambda, \mu) \to (\lambda, -\mu)$, $(\nu_i, \xi_i) \to (\nu_i, -\xi_i)$, which corresponds to the change $z \to -z$, $z_i \to -z_i$ $(i = 1, 2)$. A direct verification shows that the value $\wp(2z) - \wp(z)$ determined by the doubling formula (see [1, 2])

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z) = \left(\frac{12\lambda^2 - g_2}{4\mu}\right)^2 - 2\lambda$$

(under condition (18)) satisfies Eqs. (24) and (25). If $g_2^3 = 27g_3^2$, then curve (20) is singular, and the Weierstrass elliptic function in (21) should be replaced by its degenerate analogues (see [1])

$$\wp(z) = \frac{1}{z^2} \quad (g_2 = g_3 = 0, \omega_1 = \omega_2 = \infty),$$

$$\wp(z) = \left(\frac{\pi}{2\omega}\right)^2\left(-\frac{1}{3} + \left(\sin\frac{\pi z}{2\omega}\right)^{-2}\right)$$
$$(g_2, g_3 \neq 0, \omega_1 = \omega, \omega_2 = \infty).$$

Now, consider the case where $\mu = 0$. In this case, we have $a_{-1}a_0^2 a_1 = b_{-1}b_0^2 b_1$, and condition (18) implies $a_0(a_1 + a_{-1}) = b_0(b_1 + b_{-1})$. It follows from (27) that $a_1 = a_{-1}$ and $b_1 = b_{-1}$, i.e. $a_0 a_1 = b_0 b_1$.

Let us successively express the unknowns in terms of the parameter λ. From (23) and (26) we find $\nu_1 = \lambda - a_0$, $\nu_2 = \lambda - b_0$,

$$\xi_1^2 = 4a_0^2(3\lambda - a_0 - a_1), \quad \xi_2^2 = 4b_0^2(3\lambda - b_0 - b_1).$$

Substituting ξ_1^2 and ξ_2^2 into (30), we obtain the relations

$$12\lambda^2 - g_2 + g_3/a_0 - 4a_0a_1 = 0,$$
$$12\lambda^2 - g_2 + g_3/b_0 - 4b_0b_1 = 0,$$

which imply $g_3 = 0$ and $12\lambda^2 - g_2 - 4a_0a_1 = 0$. Moreover, according to (29), we have $4\lambda^3 - g_2\lambda = 0$. Therefore, $\lambda = 0$ or $\lambda = \pm\sqrt{a_0a_1/2}$. In each of these cases, as well as at $\mu \neq 0$, parameters (22) are determined uniquely up to the involution $(\lambda,\mu) \to (\lambda,-\mu),(\nu_i,\xi_i) \to (\nu_i,-\xi_i)$ $(i = 1,2)$.

Proof of the theorem. The proposition proved above and the relation (see [1, 2])

$$\wp(u) - \wp(v) = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)}$$

imply (5) for $-2 \leq n \leq 2$. Since the remaining terms of the sequences $\{A_n\}$ and $\{B_n\}$ are determined by relations (11)–(16) and the sequences $C_iD_i^n\dfrac{\sigma(z_i+nz)}{\sigma(z)^{n^2}}$, $i = 1,2$ satisfy the same relations (see the remark to the theorem), it follows that (5) holds for all integer n.

Corollary. *Special cases of relation (6) are*

$$\begin{vmatrix} A_{2n}B_0 & A_{n+1}B_{n-1} & A_nB_n \\ A_{1+n}B_{1-n} & A_2B_0 & A_1B_1 \\ A_nB_{-n} & A_1B_{-1} & A_0B_0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} A_{2n-1}B_0 & A_nB_{n-1} & A_{n-1}B_n \\ A_nB_{1-n} & A_1B_0 & A_0B_1 \\ A_{n-1}B_{-n} & A_0B_{-1} & A_{-1}B_0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} A_0B_{2n} & A_{n-1}B_{n+1} & A_nB_n \\ A_{1-n}B_{1+n} & A_0B_2 & A_1B_1 \\ A_{-n}B_n & A_{-1}B_1 & A_0B_0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} A_0B_{2n-1} & A_{n-1}B_n & A_nB_{n-1} \\ A_{1-n}B_n & A_0B_1 & A_1B_0 \\ A_{-n}B_{n-1} & A_{-1}B_0 & A_0B_{-1} \end{vmatrix} = 0,$$

which can also be used to calculate the elements of the sequences $\{A_n\}$ *and* $\{B_n\}$. *It follows that all terms of these sequences are Laurent polynomials in the initial data and the parameters* Δ_0, Δ_1, Δ_2:

$$A_n, B_n \in Z\left[A_0^{\pm1}, B_0^{\pm1}, A_{\pm1}, A_{\pm2}, B_{\pm1}, B_{\pm2}, \Delta_0^{\pm1}, \Delta_1^{\pm1}, \Delta_2^{\pm1}\right].$$

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation, project no. 14-11-00335.

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