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ON STATISTICAL PROPERTIES OF FINITE CONTINUED FRACTIONS

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Statistical properties of continued fractions for numbers a/b, where a and b lie in the sector $a, b \geq 1$, $a^2 + b^2 \leq R^2$, are studied. The main result is an asymptotic formula with two meaning terms for the quantity

$$N_x(R) = \sum_{\substack{a^2+b^2\leq R^2\\ a,b\in\mathbb{N}}} s_x(a/b),$$

where $s_x(a/b) = |\{j \in \{1, \dots, s\} : [0; t_j, \dots, t_s] \leq x\}|$ is the Gaussian statistic for the fraction $a/b = [t_0; t_1, \dots, t_s]$. Bibliography: 12 *titles.*

1. Notation

1. We write $[x_0; x_1, \dots, x_s]$ for a continued fraction

$$x_0 + \cfrac{1}{x_1 + \ddots + \frac{1}{x_s}}$$

of length s with formal variables x_0, $x_1, \dots, x_s$.

2. If r is a rational number, then $r = [t_0; t_1, \dots, t_s]$ stays for the canonical representation of r as a continued fraction (unless otherwise stipulated); in particular, $t_0 = [r]$ (the integer part of r), $t_1, \dots, t_s$ are positive integers, and $t_s \geq 2$ if $s \geq 1$.

3. If $x \in [0, 1]$ and $r = [t_0; t_1, \dots, t_s]$ is a rational number, then $s_x(r)$ stays for the number of indices $j \in \{1, \dots, s\}$ for which $[0; t_j, \dots, t_s] \leq x$. In particular, $s = s(r) = s_1(r)$ is the length of the above continued fraction.

4. We use the notation $K_n(x_1, \dots, x_n)$ for continuants, which are defined by the starting values

$$K_0() = 1, \quad K_1(x_1) = x_1$$

and the recurrent relation

$$K_n(x_1, \dots, x_n) = x_n K_n(x_1, \dots, x_{n-1}) + K_n(x_1, \dots, x_{n-2}) \qquad (n \geq 2).$$

Recall that we always have

$$[x_0; x_1, \dots, x_s] = \frac{K_{s+1}(x_0, x_1, \dots, x_s)}{K_s(x_1, \dots, x_s)}.$$

5. The asterisk in double sums of the form

$$\sum_n \sum_m{}^* \dots$$

means that the summation indices are connected by the additional relation $(m, n) = 1$.

6. If A is a statement, then $[A]$ is equal to 1 if A is true, and it is equal to 0 otherwise.

7. If q is a positive integer, then $\delta_q(a)$ denotes the characteristic function of divisibility by q:

$$\delta_q(a) = [a \equiv 0 \pmod p] = \begin{cases} 1 & \text{if } a \equiv 0 \pmod q, \\ 0 & \text{if } a \not\equiv 0 \pmod q. \end{cases}$$

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8. The notation for finite differences of a function $a(u,v)$ is as follows:

$$\Delta_{1,0}a(u,v) = a(u+1,v) - a(u,v), \quad \Delta_{0,1}a(u,v) = a(u,v+1) - a(u,v),$$
$$\Delta_{1,1}a(u,v) = \Delta_{0,1}(\Delta_{1,0}a(u,v)) = \Delta_{1,0}(\Delta_{0,1}a(u,v)).$$

9. The sum of powers of divisors is denoted by

$$\sigma_\alpha(q) = \sum_{d|q} d^\alpha.$$

10. The Euler's dilogarithm is defined by the relation

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\log(1-z)}{z}dz.$$

11. The Catalan constant is equal to

$$C = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \frac{1}{2i}\left[\mathrm{Li}_2(i) - \mathrm{Li}_2(-i)\right]. \tag{1}$$

2. Introduction

Some problems of the metric theory of numbers deal with statistical properties of continued fractions. For almost all real numbers α one can describe the typical behavior of partial quotients for the representation

$$\alpha = [t_0; t_1, \dots, t_s, \dots]$$

(see review [1]).

In investigating in details the Euclidean algorithm (see [6, Sec. 4.5.3]), the necessity of studying statistical properties of finite continued fractions

$$a/b = [t_0; t_1, \dots, t_s]$$

arises, provided that the numbers a and b satisfy some additional conditions. This problem was initiated by Heilbronn [9] and Dixon [7]. Heilbronn succeeded in finding the leading term in the asymptotic formula

$$\frac{1}{\varphi(b)} \sum_{\substack{1\le a\le b\\(a,b)=1}} s(a/b) = \frac{12\log 2}{\pi^2}\log b + O(1).$$

Dixon has shown that for every positive ε there exists a constant $c_0 > 0$ such that

$$\left| s(a/b) - \frac{12\log 2}{\pi^2}\log b \right| < (\log b)^{1/2+\varepsilon}$$

for all pairs (a,b) from the domain $1 \le a \le b \le R$, possibly excluding $R^2\exp(-c_0(\log R)^{\varepsilon/2})$ pairs.

Later Porter [12] has obtained a more precise result. He has shown that

$$\frac{1}{\varphi(b)} \sum_{\substack{1\le a\le b\\(a,b)=1}} s(a/b) = \frac{12\log 2}{\pi^2}\log b + C_P + O(b^{-1/6+\varepsilon}),$$

where

$$C_P = \frac{6\log 2}{\pi^2}\left(3\log 2 + 4\gamma - 24\frac{\zeta'(2)}{\pi^2} - 2\right) - \frac{1}{2}$$

is a constant, which is now known as Porter's constant (its definitive form was found by Ranch, see [10]).

The problem of studying statistical properties of continued fractions for numbers a/b with $a,\, b > 0$, $a^2+b^2 \le R^2$ was posed in book [5] (Problem 1993–11). The problem concerning the asymptotic behavior of the sum

$$N_x(R) = \sum_{\substack{a^2+b^2 \le R^2 \\ a,b\in\mathbb{N}}} s_x(a/b)$$

as $R \to \infty$ is more general. The first answer to this problem was obtained in [2]. Later in [1] a more precise asymptotic formula was proved:

$$N_x(R) = \frac{3}{\pi} \log(1+x) R^2 \log R + O(R^2);$$

the remainder term in the latter formula is better than in [2] by a term of order $\sqrt{\log R}$. In the present paper, we obtain for $N_x(R)$ an asymptotic formula with two significant terms:

$$N_x(R) = \frac{3}{\pi} R^2 \left[\log(1+x) \log R + C(x)\right] + O(R^{17/9} \log^2 R)$$

(here $C(x)$ is a function, which will be defined in the sequel).

3. Formalization of the problem

The following statement is a modification of a known theorem on continued fractions (see [3, § 50, Theorem 1]).

Lemma 1. *Let P be a nonnegative integer, and let P', Q, and Q' be positive integers such that $Q \le Q'$. Further, let α be a real number in the interval $(0;1)$. Then the following two conditions are equivalent:*

(I) *P/Q and P'/Q' are consecutive convergents of the continued fraction expansion of α, both different from α, and, moreover, the convergent P/Q precedes the convergent P'/Q';*

(II) *$PQ' - P'Q = \pm 1$ and $0 < \dfrac{Q'\alpha - P'}{-Q\alpha + P} < 1$.*

Proof. Assume that the first condition holds. The relation $PQ' - P'Q = \pm 1$ follows immediately from the properties of continued fractions. Further, since α lies between P/Q and P'/Q', there exist positive integers $t_1, \dots, t_s$ $(s \ge 1)$ and a real number α' such that $t_s < \alpha' < t_s + 1$ and

$$\frac{P}{Q} = [0; t_1, \dots, t_{s-1}], \quad \frac{P'}{Q'} = [0; t_1, \dots, t_s], \\ \alpha = [0; t_1, \dots, t_{s-1}, \alpha']. \tag{2}$$

The second condition in (II) follows from the relation

$$\frac{Q'\alpha - P'}{-Q\alpha + P} = \alpha' - t_s.$$

Let condition (II) hold. It follows from the assumptions of the lemma and the relation $|PQ' - P'Q| = 1$ that there exist positive integers $t_1, \dots, t_s$ $(s \ge 1)$ for which relations (2) are valid. Since

$$0 < \frac{Q'\alpha - P'}{-Q\alpha + P},$$

α lies between P/Q and P'/Q'; this means that there is α' such that

$$\alpha = [0; t_1, \dots, t_{s-1}, \alpha'] = \frac{(\alpha' - t_s)P + P'}{(\alpha' - t_s)Q + Q'}.$$

Replacing in the relations $0 < \dfrac{Q'\alpha - P'}{-Q\alpha + P} < 1$ the number α by the right-hand side of the above relation, we conclude that $0 < \alpha' - t_s < 1$. Therefore, $t_s = [\alpha']$, and each of the fractions P/Q and P'/Q' is a convergent of the continued fraction expansion of α.

Remark. Similarly we can prove that if $P \geq 0$, P', Q, $Q' \geq 1$, and $Q \leq Q'$, then the relations

$$PQ' - P'Q = \pm 1, \qquad \frac{Q'\alpha - P'}{-Q\alpha + P} = 1$$

are equivalent to the following fact: the fractions P/Q and P'/Q' are convergents of the form (2) for the nonstandard continued fraction expansion

$$\alpha = [0; t_1, \dots, t_{s-1}, t_s, 1] \qquad (s \geq 1)$$

of the number $\alpha = \frac{P+P'}{Q+Q'}$.

Lemma 2. *Let $R \geq 1$ and let $\Omega(R)$ be a domain on the plane Omn that is contained in the square $0 < m, n \leq R$. Assume that the boundary of the domain $\Omega(R)$ is piecewise smooth and that the length of this boundary has order $O(R)$. Denote by $M(R)$ the number of integral points that are contained in $\Omega(R)$, and by $M^*(R)$ the number of primitive points (i.e., points such that $(m,n) = 1$). Then*

$$M^*(R) = \frac{1}{\zeta(2)} M(R) + O(R \log R).$$

Proof. Let $\Omega(R/d)$ be the domain that is obtained from $\Omega(R)$ by the homothety with coefficient $1/d$ and with center at the coordinate origin. We denote by $M(R/d)$ and $M^*(R/d)$, respectively, the numbers of all integral points and of primitive points in the domain $\Omega(R/d)$, and by $V(R/d)$ the area of this domain. Applying the Möbius inversion formula (for example, see [8, Theorem 268]) to the relation

$$M(R) = \sum_{d \leq R} M^*(R/d),$$

we obtain

$$M^*(R) = \sum_{d \leq R} \mu(d) M(R/d).$$

Further, since $M(R/d) = V(R/d) + O(R/d)$ and $V(R/d) = V(R)/d^2$, we have

$$M^*(R) = \sum_{d \leq R} \mu(d) \left(\frac{V(R)}{d^2} + O\left(\frac{R}{d}\right) \right) = \frac{1}{\zeta(2)} M(R) + O(R \log R).$$

The lemma is proved.

Denote by $T_x^*(R)$ the number of solutions of the system

$$\begin{cases} PQ' - P'Q = \pm 1, \\ mP + nP' = a, \\ mQ + nQ' = b, \\ a^2 + b^2 \leq R^2 \end{cases} \tag{3}$$

such that

$$1 \leq Q \leq Q', \quad 1 \leq P' \leq Q', \quad 0 \leq P \leq Q, \quad 1 \leq m \leq xn, \quad (m,n) = 1. \tag{4}$$

Lemma 3. *For every $R \geq 2$ and for $x \in [0;1]$ the following relation holds:*

$$N_x^*(R) = 2T_x^*(R) + \frac{\pi}{2\zeta(2)} R^2 \arctan x \cdot (1 - 2[x = 1]) + O(R \log R). \tag{5}$$

Proof. Let a/b be a fixed rational number in the interval $(0;1)$; we take the irreducible representation of this number, i.e., we assume that $(a,b) = 1$. Expand it into the continued fraction

$$a/b = [0; t_1, t_2, \dots, t_{s-1}, t_s] \qquad (s \geq 1).$$

We shall study the quantity $s_x(a/b)$ that is defined as the number of indices $j \in \{1,2,\dots,s\}$ such that $[0;t_j,\dots,t_s] \le x$, where x is a fixed real number in the interval $[0;1]$.

Let $s \ge 2$, and let P/Q and P'/Q' be consecutive convergents of the continued fraction expansion of a/b (the fraction P/Q precedes the fraction P'/Q'); we assume that both fractions are different from a/b. Then for a certain index $j \in \{1,2,\dots,s-1\}$ we have

$$P = K_{j-2}(t_2,\dots,t_{j-1}), \qquad P' = K_{j-1}(t_2,\dots,t_j),$$
$$Q = K_{j-1}(t_1,\dots,t_{j-1}), \qquad Q' = K_j(t_1,\dots,t_j)$$

(in particular, if $j=1$, then $P=0$, $Q=P'=K()=1$, $Q'=t_1$). Since $PQ'-P'Q=\pm 1$, for the pair of integers a, b there exist unique integers m, n such that

$$mP + nP' = a,$$
$$mQ + nQ' = b.$$

It follows from the properties of continuants (for example, see [4]) that the numbers

$$m = K_{s-j-1}(t_{j+2},\dots,t_s),$$
$$n = K_{s-j}(t_{j+1},\dots,t_s)$$

satisfy the above equations; moreover, $m/n = [0;t_{j+1},\dots,t_s]$.

By Lemma 1,

$$s_x(a/b) = [a/b \le x] + l_x(a,b),$$

where $l_x(a,b)$ is the number of solutions of the system

$$\begin{cases} PQ'-P'Q=\pm 1, \\ 0 < \dfrac{aQ'-bP'}{-aQ+bP} < 1, \\ mP+nP'=a, \\ mQ+nQ'=b, \end{cases} \tag{6}$$

$$1 \le Q \le Q', \quad 1 \le P' \le Q', \quad 0 \le P \le Q, \quad m/n \le x.$$

Further, since

$$\frac{aQ'-bP'}{-aQ+bP} = \frac{m}{n},$$

we can rewrite system (6) in the form

$$\begin{cases} PQ'-P'Q=\pm 1, \\ mP+nP'=a, \\ mQ+nQ'=b, \end{cases}$$

$$1 \le Q \le Q', \quad 1 \le P' \le Q', \quad 0 \le P \le Q, \quad m/n \le x, \quad 0 < m < n.$$

Since $b/a = [t_1;t_2,\dots,t_{s-1},t_s]$, we have

$$s_x(b/a) = s_x(a/b) - [a/b \le x] = l_x(a,b),$$
$$s_x(b/a) + s_x(a/b) = 2l_x(a,b) + [a/b \le x]. \tag{7}$$

Summation of formula (7) over all primitive points (a,b) that lie in the sector

$$\{(a,b) : 1 \le a \le b, a^2+b^2 \le R^2\}$$

yields the relation

$$N_x^*(R) = 2L_x^*(R) + \frac{\pi}{2\zeta(2)}\, R^2 \arctan x + O\left(R \log R\right), \tag{8}$$

where $L_x^*(R)$ is the number of solutions of system (3) for which

$$1 \leq Q \leq Q', \quad 1 \leq P' \leq Q', \quad 0 \leq P \leq Q,$$
$$0 < m < n, \quad m/n \leq x, \quad (m,n) = 1.$$

If $x < 1$ or $n \geq 2$, then we can ignore the requirement $m < n$. In this case, $L_x^*(R) = T_x^*(R)$ and the lemma is proved. If $x = 1$ and $m = n = 1$, then the elimination of the requirement $m < n$ increases the number of solutions of system (3). Therefore,

$$L_x^*(R) = T_x^*(R) - T_0, \tag{9}$$

where T_0 is the number of solutions of the system

$$\begin{cases} PQ' - P'Q = \pm 1, \\ P + P' = a, \\ Q + Q' = b, \\ a^2 + b^2 \leq R^2, \end{cases} \tag{10}$$

$$1 \leq Q \leq Q', \quad 1 \leq P' \leq Q', \quad 0 \leq P \leq Q.$$

By the remark to Lemma 1, for every primitive point (a,b) such that $1 \leq a < b$, system (10) has exactly one solution. Hence, by Lemma 2,

$$T_0 = \frac{\pi}{2\zeta(2)} R^2 \arctan 1 + O\left(R \log R\right). \tag{11}$$

Formulas (8), (9), and (11) imply the lemma.

To study the quantity $T_x^*(R)$, we introduce a new parameter U, which lies in the interval $1 \leq U \leq R$. By T_1 we denote the number of solutions of system (3) with restrictions (4), which satisfy the additional condition $Q' \leq U$. The number of solutions for which $Q' > U$ will be denoted by T_2. Then

$$T_x^*(R) = T_1 + T_2.$$

We shall study the numbers T_1 and T_2 separately.

4. Evaluation of the number T_1

Lemma 4. *Let $q \geq 1$ be an integer, Q_1, Q_2, P_1, P_2 be real numbers, and $0 \leq P_1, P_2 \leq q$. Then the number*

$$\Phi_q(Q_1, Q_2; P_1, P_2) = \sum_{\substack{Q_1 < u \leq Q_1 + P_1 \\ Q_2 < v \leq Q_2 + P_2}} \delta_q(uv - 1)$$

satisfies the asymptotic relation

$$\Phi_q(Q_1, Q_2; P_1, P_2) = \frac{\varphi(q)}{q^2} P_1 P_2 + O\left(\psi(q)\sqrt{q}\right)$$

in which $\psi(q) = \sigma_0(q)\sigma_{-1/2}(q)\log^2(q+1)$.

For a proof, see [1].

Lemma 5. *Let $q \geq 1$ be an integer, and let $a(u,v)$ be a function that is defined at integral points (u,v) such that $1 \leq u, v \leq q$. Assume that this function satisfies the inequalities*

$$a(u,v) \geq 0, \quad \Delta_{1,0} a(u,v) \leq 0, \quad \Delta_{0,1} a(u,v) \leq 0, \quad \Delta_{1,1} a(u,v) \geq 0 \tag{12}$$

at all points at which these conditions have meaning. Then the sum

$$W = \sum_{u,v=1}^{q} \delta_q(uv - 1) a(u,v)$$

satisfies the asymptotic relation

$$W = \frac{\varphi(q)}{q^2} \sum_{u,v=1}^{q} a(u,v) + O\left(A\psi(q)\sqrt{q}\right),$$

in which $\psi(q)$ is the function from Lemma 4 and $A = a(1,1)$ is the maximum of the function $a(u,v)$.

Proof. Extend the function $a(u,v)$ to a larger domain by setting

$$a(u,q+1) = a(q+1,v) = 0 \qquad (1 \le u,v \le q+1).$$

Then it follows from inequalities (12) that $\Delta_{1,1}a(u,v) \ge 0$ for all integers u and v such that $1 \le u,v \le q$.

Apply the Abel transform

$$\sum_{n=1}^{q} f(n)g(n) = g(q+1)\sum_{n=1}^{q} f(n) - \sum_{k=1}^{q}\left(\sum_{n=1}^{k} f(n)\right)(g(k+1)-g(k))$$

to the sum W, first with respect to the variable u and then with respect to the variable v. Setting first $f(u) = \delta_q(uv-1)$, $g(u) = a(u,v)$ and then $f(v) = \sum\limits_{u=1}^{k} \delta_q(uv-1)$, $g(u) = \Delta_{1,0}a(u,v)$, we obtain

$$W = \sum_{k,l=1}^{q} \Delta_{1,1}a(k,l) \sum_{u=1}^{k}\sum_{v=1}^{l} \delta_q(uv-1).$$

By Lemma 4, the inner double sum satisfies the asymptotic formula

$$\sum_{u=1}^{k}\sum_{v=1}^{l} \delta_q(uv-1) = \frac{\varphi(q)}{q^2}kl + O\left(\psi(q)\sqrt{q}\right).$$

Therefore,

$$W = \frac{\varphi(q)}{q^2} \sum_{k,l=1}^{q} \Delta_{1,1}a(k,l)\, kl + O\left(\psi(q)\sqrt{q} \sum_{k,l=1}^{q} |\Delta_{1,1}a(k,l)|\right).$$

Since we always have $|\Delta_{1,1}a(k,l)| = \Delta_{1,1}a(k,l)$, we obtain

$$W = \frac{\varphi(q)}{q^2} \sum_{k,l=1}^{q} \Delta_{1,1}a(k,l) \sum_{u=1}^{k}\sum_{v=1}^{l} 1 + O\left(A\psi(q)\sqrt{q}\right).$$

Changing the order of summation so that the summation over u and v becomes outer, and summing over k and l, we obtain the lemma.

Lemma 6. *Let $q \ge 1$ be an integer and $x \in [0;1]$. Then the sum*

$$W_1(q) = \sum_{u,v=1}^{q} \delta_q(uv-1)\left[\arctan\frac{u}{q} - \arctan\left(\frac{u}{q} - \frac{x}{q(q+vx)}\right)\right]$$

satisfies the asymptotic formula

$$W_1(q) = \frac{\pi}{4}\log(1+x)\frac{\varphi(q)}{q^2} + O\left(\frac{\psi(q)}{q^{3/2}}\right).$$

Proof. Using the Lagrange intermediate value theorem, we verify that the function

$$a(u,v) = \arctan\frac{u}{q} - \arctan\left(\frac{u}{q} - \frac{x}{q(q+vx)}\right)$$

satisfies conditions (12). Therefore, by Lemma 5,

$$W_1(q) = \frac{\varphi(q)}{q^2} \sum_{u,v=1}^{q} \left[\arctan \frac{u}{q} - \arctan \left(\frac{u}{q} - \frac{x}{q(q+vx)} \right) \right] + O\left(\frac{\psi(q)\sqrt{q}}{q^2} \right).$$

Applying the Lagrange theorem once again, we obtain

$$\arctan \frac{u}{q} - \arctan \left(\frac{u}{q} - \frac{x}{q(q+vx)} \right) = \frac{x}{q(q+vx)} \cdot \frac{1}{1+\frac{u^2}{q^2}} \left(1 + O\left(\frac{1}{q^2} \right) \right),$$

$$\frac{x}{q+vx} = \log(q+vx) - \log(q+(v-1)x) + O\left(\frac{1}{q^2} \right),$$

$$\frac{1}{q\left(1+\frac{u^2}{q^2}\right)} = \arctan \frac{u}{q} - \arctan \frac{u-1}{q} + O\left(\frac{1}{q^2} \right),$$

Hence,

$$\begin{aligned} W_1(q) = \frac{\varphi(q)}{q^2} \sum_{u=1}^{q} &\left(\arctan \frac{u}{q} - \arctan \frac{u-1}{q} \right) \\ &\sum_{v=1}^{q} [\log(q+vx) - \log(q+(v-1)x)] \\ +O\left(\frac{\psi(q)}{q^{3/2}} \right) &= \frac{\pi}{4} \log(1+x) \frac{\varphi(q)}{q^2} + O\left(\frac{\psi(q)}{q^{3/2}} \right). \end{aligned}$$

Corollary 1. *If $N \ge 1$, then the sum*

$$W_2 = \sum_{q \le N} W_1(q)$$

satisfies the asymptotic relation

$$W_2 = \frac{\pi}{4} \cdot \frac{\log(1+x)}{\zeta(2)} \left(\log N + \gamma - \frac{\zeta'(2)}{\zeta(2)} \right) + f(x) + O\left(\frac{\log^5(N+1)}{\sqrt{N}} \right), \tag{13}$$

in which $f(x)$ is the sum of the (infinite) series

$$f(x) = \sum_{q=1}^{\infty} \left(W_1(q) - \frac{\pi}{4} \log(1+x) \frac{\varphi(q)}{q^2} \right). \tag{14}$$

Proof. Using the estimate $\psi(q) \le \sigma_0^2(q) \log^2(q+1)$ and the Abel transform, we obtain

$$\sum_{q>N} \frac{\psi(q)}{q^{3/2}} = O\left(\frac{\log^5(N+1)}{\sqrt{N}} \right).$$

Therefore,

$$\sum_{q \le N} \left(W_1(q) - \frac{\pi}{4} \log(1+x) \frac{\varphi(q)}{q^2} \right) = f(x) + O\left(\frac{\log^5(N+1)}{\sqrt{N}} \right),$$

$$W_2 = \frac{\pi}{4} \cdot \log(1+x) \sum_{q \le N} \frac{\varphi(q)}{q^2} + f(x) + O\left(\frac{\log^5(N+1)}{\sqrt{N}} \right). \tag{15}$$

Expressing $\varphi(q)$ in terms of the Möbius function, we have

$$\sum_{q\le N}\frac{\varphi(q)}{q^2}=\sum_{q\le N}\frac{1}{q}\sum_{d|q}\frac{\mu(d)}{d}=\sum_{d\le N}\frac{\mu(d)}{d^2}\sum_{q\le N/d}\frac{1}{q}$$

$$=\sum_{d\le N}\frac{\mu(d)}{d^2}\left(\log N-\log d+\gamma+O\left(\frac{d}{N}\right)\right).$$

Since

$$\sum_{d\le N}\frac{\mu(d)}{d^2}=\frac{1}{\zeta(2)}+O\left(\frac{1}{N}\right),$$

$$\sum_{d\le N}\frac{\mu(d)}{d^2}\log d=\frac{\zeta'(2)}{\zeta^2(2)}+O\left(\frac{\log(N+1)}{N}\right),$$

we have

$$\sum_{q\le N}\frac{\varphi(q)}{q^2}=\frac{1}{\zeta(2)}\left(\log N+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\log(N+1)}{N}\right).$$

Substituting the latter formula into relation (15), we complete the proof of the corollary.

Remark. Similarly we can check that for the sum

$$W_3(q)=\sum_{u,v=1}^{q}\delta_q(uv+1)\left[\arctan\left(\frac{u}{q}+\frac{x}{q(q+vx)}\right)-\arctan\frac{u}{q}\right]$$

the asymptotic formula

$$W_3(q)=\frac{\pi}{4}\log(1+x)\frac{\varphi(q)}{q^2}+O\left(\frac{\psi(q)}{q^{3/2}}\right)$$

is true and that for $N\ge 1$ the sum

$$W_4=\sum_{q\le N}W_3(q)$$

can be represented in the form

$$W_4=\frac{\pi}{4}\cdot\frac{\log(1+x)}{\zeta(2)}\left(\log N+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+g(x)+O\left(\frac{\log^5(N+1)}{\sqrt{N}}\right), \tag{16}$$

where $g(x)$ is the function given by its expansion

$$g(x)=\sum_{q=1}^{\infty}\left(W_3(q)-\frac{\pi}{4}\log(1+x)\frac{\varphi(q)}{q^2}\right). \tag{17}$$

Theorem 1. *Let $1\le U\le R$. Then the number T_1 of solutions of system* (3), (4) *with the additional restriction $Q'\le U$ satisfies the asymptotic formula*

$$T_1=\frac{\pi}{4}\cdot\frac{R^2}{\zeta^2(2)}\left[\log(x+1)\left(\log U+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+C_1(x)\right]$$

$$+O\left(R^2U^{-1/2}\log^5 R\right)+O\left(RU\log R\right),$$

in which

$$C_1(x)=\frac{2}{\pi}\left(f(x)+g(x)-\arctan\frac{x}{2+3x}\right) \tag{18}$$

and $f(x)$ and $g(x)$ are functions defined by relations (14) and (17).

Proof. If the values of the parameters P, P', Q, and Q' are fixed, then the number of solutions of system (3) with unknown variables m and n is equal to the number of primitive points (m,n) in the domain $\Omega=\Omega(P,P',Q,Q')$ that is defined by the conditions

$$(mP+nP')^2+(mQ+nQ')^2\le R^2,\quad 1\le m\le nx.$$

This domain is contained in the square $0<m,n\le R/Q'$, its boundary is piecewise smooth, and the length of the boundary is equal to $O(R/Q')$. By Lemma 2, the number of such points is equal to

$$\frac{1}{\zeta(2)}V(\Omega)+O\left(\frac{R}{Q'}\log R\right).$$

It follows that

$$T_1=\frac{1}{\zeta(2)}\sum_{PQ'-P'Q=\pm1}V(\Omega)+O\left(RU\log R\right),$$

where summation proceeds over the quadruples (P,P',Q,Q') that satisfy restrictions (4) and the condition $Q'\le U$. Replacing the variables m and n by the initial parameters a and b

$$m=\pm(aQ'-bP'),\quad n=\pm(bP-aQ),$$

we conclude that the number $V(\Omega)$ coincides with the area of the sector

$$\frac{aQ'-bP'}{bP-aQ}\le x,\quad \pm(aQ'-bP')>0,\quad a^2+b^2\le R^2$$

on the plane Oab. Therefore,

$$V(\Omega)=\pm\frac{R^2}{2}\left(\arctan\frac{P'+Px}{Q'+Qx}-\arctan\frac{P'}{Q'}\right),$$

where the sign before the bracket is the same as that on the right-hand side of the relation $PQ'-P'Q=\pm1$. If the value of the parameter $Q'=q$ is fixed, the variables P' and Q necessarily satisfy the congruence $P'Q\pm1\equiv0 \pmod q$. If $P'=u$, $Q=v$ is a solution of such a congruence, then the value of the parameter P is uniquely determined: $P=(uv\pm1)/q$. The area of the sector, depending on the choice of the sign, is equal to

$$\frac{R^2}{2}\left[\arctan\frac{u}{q}-\arctan\left(\frac{u}{q}-\frac{x}{q(q+vx)}\right)\right]$$

or

$$\frac{R^2}{2}\left[\arctan\left(\frac{u}{q}+\frac{x}{q(q+vx)}\right)-\arctan\frac{u}{q}\right].$$

The value of the parameter P always falls in the required limits $0\le P\le Q=v$, with the only exception $q=u=v=1$, $P=2$. Hence,

$$\begin{aligned}T_1=&\frac{R^2}{2\zeta(2)}\sum_{q\le U}\sum_{u,v=1}^{q}\delta_q(uv-1)\left[\arctan\frac{u}{q}-\arctan\left(\frac{u}{q}-\frac{x}{q(q+vx)}\right)\right]\\&+\frac{R^2}{2\zeta(2)}\sum_{q\le U}\sum_{u,v=1}^{q}\delta_q(uv+1)\left[\arctan\left(\frac{u}{q}+\frac{x}{q(q+vx)}\right)-\arctan\frac{u}{q}\right]\\&-\frac{R^2}{2\zeta(2)}\left[\arctan\left(1+\frac{x}{x+1}\right)-\arctan 1\right]+O\left(RU\log R\right).\end{aligned}$$

Replacing in this formula the sums W_2 and W_4 by their asymptotic values (13) and (16) and observing that

$$\arctan\left(1+\frac{x}{x+1}\right)-\arctan 1=\arctan\frac{x}{2+3x},$$

we complete the proof of the theorem.

5. Evaluation of the number T_2

Lemma 7. *Let $q \geq 1$, and let $f(u)$ be a nonnegative nonincreasing function on the segment $[0; q]$ such that $f(0) \leq q$. Then*

$$\sum_{u=1}^{q} \sum_{1 \leq v \leq f(u)} \delta_q(uv \pm 1) = \frac{\varphi(q)}{q^2} V(\Omega) + O\left(q^{3/4}\sigma_0(q)\log(q+1)\right),$$

where Ω is the domain on the plane Ouv defined by the conditions $0 \leq u \leq q$ and $0 \leq v \leq f(u)$.

Proof. Divide the interval of summation over the variable u into k parts $(1 \leq k \leq q)$:

$$0 = u_0 < u \leq u_1, \dots, u_{k-1} < u \leq u_k = q,$$

where $u_j = jq/k$. Set

$$S = \sum_{j=1}^{k} S_j, \qquad S_j = \sum_{u_{j-1} < u \leq u_j} \sum_{1 \leq v \leq f(u)} \delta_q(uv \pm 1).$$

Since the function $f(u)$ is monotone, we have

$$\sum_{u_{j-1} < u \leq u_j} \sum_{1 \leq v \leq f(u_j)} \delta_q(uv \pm 1) \leq S_j \leq \sum_{u_{j-1} < u \leq u_j} \sum_{1 \leq v \leq f(u_{j-1})} \delta_q(uv \pm 1).$$

Applying Lemma 4 to the sums in the latter formula, we obtain the inequalities

$$\frac{\varphi(q)}{q^2} \cdot \frac{q}{k} f(u_j) + O\left(\sqrt{q}\psi(q)\right) \leq S_j \leq \frac{\varphi(q)}{q^2} \cdot \frac{q}{k} f(u_{j-1}) + O\left(\sqrt{q}\psi(q)\right). \tag{19}$$

Since

$$\frac{q}{k} \sum_{j=1}^{k} f(u_j) = \int_0^q f(u)\, du + O\left(\frac{q}{k} f(0)\right),$$

$$\frac{q}{k} \sum_{j=0}^{k-1} f(u_j) = \int_0^q f(u)\, du + O\left(\frac{q}{k} f(0)\right),$$

and $f(0) \leq q$, summation over j of estimates (19) provides the asymptotic formula

$$S = \frac{\varphi(q)}{q^2} \int_0^q f(u)\, du + O\left(\frac{q}{k}\right) + O\left(k\sqrt{q}\psi(q)\right).$$

Setting

$$k = q^{1/4}(\sigma_0(q)\log(q+1))^{-1},$$

we complete the proof of the lemma.

Lemma 8. *Let $1 \leq U < R$ and $R_1 = R/U$. Then the number T_2 of solutions of system* (3), (4) *with the additional restriction $Q' > U$ satisfies the asymptotic formula*

$$T_2 = 2 \sum_{n < R_1} \sideset{}{^*}\sum_{m \leq nx} \sum_{U < q \leq R} \frac{\varphi(q)}{q^2} V(m, n, q) + O\left(R^2 U^{-1/4} \log^2 R\right),$$

in which $V(m, n, q)$ is the area of the domain $\Omega(m, n, q)$ on the plane Ouv defined by the conditions

$$0 \leq u, v \leq q, \quad \left(\frac{u^2}{q^2} + 1\right)(mv + nq)^2 \leq R^2.$$

Proof. It follows from the definition of the number T_2 that

$$T_2 = \sum_{2\le n<R_1} \sideset{}{^*}\sum_{m\le nx} \sum_{U<q\le R/n} \sum_{u,v=1}^{q} \delta_q(uv\pm 1)\left[\left(m\frac{uv\pm 1}{q}+nu\right)^2+(mv+nq)^2\le R^2\right].$$

By Lemma 7,

$$T_2 = \sum_{2\le n<R_1} \sideset{}{^*}\sum_{m\le nx} \sum_{U<q\le R/n} \left(\frac{\varphi(q)}{q^2}V_\pm(m,n,q)+O\left(q^{3/4}\sigma_0(q)\log q\right)\right),$$

where $V_\pm(m,n,q)$ is the area of the domain $\Omega_\pm(m,n,q)$ on the plane Ouv that is defined by the conditions

$$0\le u,v\le q,\quad \left(m\frac{uv\pm 1}{q}+nu\right)^2+(mv+nq)^2\le R^2.$$

Since

$$\Omega_+(m,n,q)\subset\Omega(m,n,q)\subset\Omega_-(m,n,q),$$

the replacement of $V_\pm(m,n,q)$ by $V(m,n,q)$ leads to an error that does not exceed the difference $V_-(m,n,q)-V_+(m,n,q)$. But if u is fixed, the difference between the numbers v_- and v_+ satisfying the relation

$$\left(m\frac{uv_\pm\pm 1}{q}+nu\right)^2+(mv_\pm+nq)^2=R^2$$

has order $O(u/q^2)$. Therefore, $V_-(m,n,q)-V_+(m,n,q)=O(1)$ and

$$T_2 = 2\sum_{n<R_1} \sideset{}{^*}\sum_{m\le nx} \sum_{U<q\le R/n} \left[\frac{\varphi(q)}{q^2}V(m,n,q)+O\left(q^{3/4}\sigma_0(q)\log q\right)\right].$$

Summing the remainder terms, we obtain

$$T_2 = 2\sum_{n<R_1} \sideset{}{^*}\sum_{m\le nx} \sum_{U<q\le R/n} \frac{\varphi(q)}{q^2}V(m,n,q)+O\left(R^2U^{-1/4}\log^2 R\right).$$

It remains to note that the requirement $q\le R/n$ can be replaced by an easier requirement $q<R$, because the domain $\Omega(m,n,q)$ is void and $V(m,n,q)=0$, provided that $nq>R$.

Lemma 9. *Let $1\le U\le R$ and $R_1=R/U$. Then the sum*

$$W_5=\sum_{n<R_1} \sideset{}{^*}\sum_{m\le nx} \sum_{U<q\le R} \frac{\varphi(q)}{q^2}V(m,n,q)$$

satisfies the asymptotic formula

$$W_5=\frac{U^2}{\zeta(2)}\int_0^1 dt\int_0^{R_1(t)}\xi\, F^*(\xi)\,d\xi+O(R^2U^{-1}\log R),$$

in which $R_1(t)=R_1/\sqrt{t^2+1}$ and

$$F^*(\xi)=\sum_{n<\xi} \sideset{}{^*}\sum_{m\le nx}\frac{1}{m}\left(\frac{1}{n}-\frac{1}{m+n}\right)[\xi\ge m+n]+\sum_{n<\xi} \sideset{}{^*}\sum_{m\le nx}\frac{1}{m}\left(\frac{1}{n}-\frac{1}{\xi}\right)[\xi<m+n].$$

Proof. First we find an approximate value of the sum

$$\sum_{U<q\le R}\frac{\varphi(q)}{q^2}V(m,n,q).$$

Represent the number $V(m,n,q)$ as an integral:

$$V(m,n,q)=\int_0^q du\int_0^q dv\left[\sqrt{u^2/q^2+1}(mv+nq)\le R\right].$$

Introduce new variables $t=u/q$ and $w=mv+nq$ and a new function $R(t)=R/\sqrt{t^2+1}$; in this notation,

$$V(m,n,q)=\frac{q}{m}\int_0^1 dt\int_0^{R(t)} dw\left[\frac{w}{m+n}<q\le\frac{w}{n}\right].$$

Further,

$$\sum_{U<q\le R}\frac{\varphi(q)}{q^2}V(m,n,q)=\sum_{\delta}\frac{\mu(\delta)}{\delta^2}\sum_{\frac{U}{\delta}<q\le\frac{R}{\delta}}\frac{V(m,n,\delta q)}{q}.$$

Evaluate the inner sum:

$$\sum_{\frac{U}{\delta}<q\le\frac{R}{\delta}}\frac{V(m,n,\delta q)}{q}=\frac{\delta}{m}\int_0^1 dt\int_0^{R(t)} dw\sum_{\frac{U}{\delta}<q\le\frac{R}{\delta}}\left[\frac{w}{m+n}<q\le\frac{w}{n}\right]$$

$$=\frac{\delta}{m}\int_0^1 dt\int_0^{R(t)} dw\left(\frac{w}{n\delta}-\max\left\{\frac{w}{(m+n)\delta},\frac{U}{\delta}\right\}+O(1)\right)[w\ge nU]$$

$$=\frac{1}{m}\int_0^1 dt\int_0^{R(t)} dw\left(\frac{w}{n}-\max\left\{\frac{w}{m+n},U\right\}\right)[w\ge nU]+O\left(\frac{\delta R}{m}\right).$$

Set $\xi=w/U$; we have

$$\sum_{\frac{U}{\delta}<q\le\frac{R}{\delta}}\frac{\varphi(q)}{q^2}V(m,n,q)=\frac{U^2}{m}\int_0^1 dt\int_0^{R_1(t)} d\xi\left(\frac{\xi}{n}-\max\left\{\frac{\xi}{m+n},1\right\}\right)[\xi\ge n]+O\left(\frac{\delta R}{m}\right).$$

Hence,

$$\sum_{U<q\le R}\frac{\varphi(q)}{q^2}V(m,n,q)$$

$$=\frac{U^2}{m\zeta(2)}\int_0^1 dt\int_0^{R_1(t)} d\xi\left(\frac{\xi}{n}-\max\left\{\frac{\xi}{m+n},1\right\}\right)[\xi\ge n]+O\left(\frac{R}{m}\log R\right)$$

$$=\frac{U^2}{m\zeta(2)}\int_0^1 dt\int_0^{R_1(t)} d\xi\left(\frac{\xi}{n}-\frac{\xi}{m+n}\right)[\xi\ge m+n]$$

$$+\frac{U^2}{m\zeta(2)}\int_0^1 dt\int_0^{R_1(t)} d\xi\left(\frac{\xi}{n}-1\right)[n\le\xi\le m+n]+O\left(\frac{R}{m}\log R\right).$$

Summing the latter relation over n and m, we obtain the lemma.

Corollary 2. *Let $1 \le U \le R$, $R_1 = R/U$, and $R_1(t) = R_1/\sqrt{t^2+1}$. Then we have the following asymptotic formula for the number T_2:*

$$T_2 = 2\frac{U^2}{\zeta(2)}\int\limits_0^1 dt \int\limits_0^{R_1(t)} \xi\, F^*(\xi)\, d\xi + O(R^2 U^{-1/4}\log^2 R).$$

Indeed, this statement follows immediately from Lemmas 8 and 9.

Lemma 10. *If $N > 1$, then the sum*

$$F^*(N) = \sum_{n<N}\sum_{m\le nx}{}^{*} \frac{1}{m}\left(\frac{1}{n} - \frac{1}{m+n}\right) - \sum_{n<N}\sum_{\substack{m\le nx\\ m+n>N}}{}^{*} \frac{1}{m}\left(\frac{1}{N} - \frac{1}{m+n}\right)$$

satisfies the asymptotic formula

$$F^*(N) = \frac{\log(x+1)}{\zeta(2)}\log N + \frac{H(x)}{\zeta(2)} + O\left(\frac{\log^2(N+1)}{N}\right),$$

in which

$$H(x) = \log(x+1)\left(\log x - \frac{\zeta'(2)}{\zeta(2)} - \frac{1}{2}\log(x+1) + \gamma - 1\right) + h(x)$$

and

$$h(x) = \sum_{m=1}^{\infty}\left(\sum_{\frac{m}{x}\le n<\frac{m}{x}+m}\frac{1}{n} - \log(x+1)\right). \tag{20}$$

Proof. We begin with the study of the sum $F(N)$, which differs from the sum $F^*(N)$ by the absence of the requirement that m and n be relatively prime in the inner summation. Set $F(N) = F_1(N) - F_2(N)$, where

$$F_1(N) = \sum_{n<N}\sum_{m\le nx}\frac{1}{m}\left(\frac{1}{n} - \frac{1}{m+n}\right),$$

$$F_2(N) = \sum_{n<N}\sum_{\substack{m\le nx\\ m+n>N}}\frac{1}{m}\left(\frac{1}{N} - \frac{1}{m+n}\right).$$

Using the function $h(x)$, which was introduced above, we obtain

$$\begin{aligned} F_1(N) &= \sum_{m<xN}\frac{1}{m}\left(\sum_{\frac{m}{x}\le n<\frac{m}{x}+m}\frac{1}{n} - \sum_{N\le n<N+m}\frac{1}{n}\right) \\ &= h(x) + \log(x+1)\sum_{m<xN}\frac{1}{m} - \sum_{m<xN}\frac{1}{m}\sum_{N\le n<N+m}\frac{1}{n} + O\left(\frac{1}{N}\right) \\ &= h(x) + (\log(x+1) + \log N)(\log xN + \gamma) - \sigma + O\left(\frac{\log(N+1)}{N}\right), \end{aligned}$$

where

$$\sigma = \sum_{m<xN}\frac{\log(N+m)}{m}. \tag{21}$$

Represent the number $F_2(N)$ in the form $F_2(N) = F_3(N) - F_4(N)$, where

$$F_3(N) = \frac{1}{N}\sum_{n<N}\sum_{\substack{m\le nx\\ m+n>N}}\frac{1}{m},$$

$$F_4(N) = \sum_{n<N} \sum_{\substack{m \le nx \\ m+n>N}} \frac{1}{m} \cdot \frac{1}{m+n}.$$

Changing the order of summation in the sum $F_3(N)$, we derive that

$$F_3(N) = \frac{1}{N} \sum_{m \le \frac{xN}{x+1}} \frac{1}{m} \sum_{N-m<n<N} 1 + \frac{1}{N} \sum_{\frac{xN}{x+1}<m<xN} \frac{1}{m} \sum_{\frac{m}{x} \le n<N} 1$$

$$= \log(x+1) + O\left(\frac{\log(N+1)}{N}\right).$$

Similarly,

$$F_4(N) = \sum_{m \le \frac{xN}{x+1}} \frac{1}{m} \sum_{N-m<n<N} \frac{1}{m+n} + \sum_{\frac{xN}{x+1}<m<xN} \frac{1}{m} \sum_{\frac{m}{x} \le n<N} \frac{1}{m+n}$$

$$= \sigma - \log N(\log xN + \gamma) - \frac{1}{2}\log^2(x+1) + O\left(\frac{\log(N+1)}{N}\right),$$

where the number σ is defined by relation (21). Hence,

$$F_2(N) = \log N(\log xN + \gamma) + \frac{\log^2(x+1)}{2} + \log(x+1) - \sigma + O\left(\frac{\log(N+1)}{N}\right),$$

$$F(N) = \log(x+1)\left(\log Nx - \frac{\log(x+1)}{2} + \gamma - 1\right) + h(x) + O\left(\frac{\log(N+1)}{N}\right).$$

Using the Möbius inversion formula, we obtain

$$F^*(N) = \sum_{d<N} \frac{\mu(d)}{d^2} F(N/d) = \frac{\log(x+1)}{\zeta(2)}\left(\log Nx - \frac{\zeta'(2)}{\zeta(2)} - \frac{\log(x+1)}{2} + \gamma - 1\right)$$
$$+\frac{h(x)}{\zeta(2)} + O\left(\frac{\log^2(N+1)}{N}\right).$$

Lemma 11. *Let $X > 0$ and $X(t) = X/\sqrt{t^2+1}$. Then*

$$\int_0^1 dt \int_0^{X(t)} \xi \, d\xi = \frac{\pi}{8} X^2,$$

$$\int_0^1 dt \int_0^{X(t)} \xi \log \xi \, d\xi = \frac{\pi}{8} X^2 \left(\log\frac{X}{2} + 2\frac{C}{\pi} - \frac{1}{2}\right),$$

where C is the Catalan constant defined by relations (1).

Proof. Denote the integrals in the statement by I_1 and I_2, respectively. Make the change of variable $y = \xi^2$ in the first integral; we obtain

$$I_1 = \frac{1}{2}\int_0^1 dt \int_0^{X^2(t)} dy = \frac{X^2}{2}\int_0^1 \frac{dt}{t^2+1} = \frac{X^2}{2} \arctan t \Big|_{t=0}^{1} = \frac{\pi}{8} X^2.$$

To prove the second relation, we first evaluate the integral

$$I_0 = \int_0^1 \frac{\log(t^2+1)}{t^2+1} dt.$$

Consider the principal branch of the logarithm $\log z$ for which $|\arg z| < \pi$. The formula

$$\mathrm{Li}_2(z) = -\int\limits_0^z \frac{\log(1-t)}{t}dt$$

determines the principal branch of the dilogarithm, which is defined on the whole complex plane, except for the ray $[1; +\infty)$. We can give explicitly the antiderivative of the function in the integral I_0:

$$\int \frac{\log(t^2+1)}{t^2+1}dt = \frac{\arctan t}{2}[\log(t^2+1) + 2\log 2] + \frac{i}{2}\left[\mathrm{Li}_2\left(\frac{1+it}{2}\right) - \mathrm{Li}_2\left(\frac{1-it}{2}\right)\right].$$

Therefore,

$$I_0 = \int\limits_0^1 \frac{\log(t^2+1)}{t^2+1}dt = \frac{3\pi}{8}\log 2 + \frac{i}{2}\left[\mathrm{Li}_2\left(\frac{1+i}{2}\right) - \mathrm{Li}_2\left(\frac{1-i}{2}\right)\right].$$

Using now the identity (see [11])

$$\mathrm{Li}_2\left(\frac{z}{z-1}\right) = -\mathrm{Li}_2(z) - \frac{1}{2}\log^2(1-z), \qquad z \notin [1; +\infty),$$

for $z = \frac{1\pm i}{2}$ we derive that

$$\frac{i}{2}\left[\mathrm{Li}_2\left(\frac{1+i}{2}\right) - \mathrm{Li}_2\left(\frac{1-i}{2}\right)\right] = \frac{i}{2}\left[\mathrm{Li}_2(i) - \mathrm{Li}_2(-i)\right] + \frac{\pi}{8}\log 2 = -C + \frac{\pi}{8}\log 2.$$

Thus,

$$I_0 = -C + \frac{\pi}{2}\log 2.$$

We make the change of variable $y = \xi^2$ in the second integral; we obtain

$$I_2 = \frac{1}{4}\int\limits_0^1 dt \int\limits_0^{X^2(t)} \log y\, dy = \frac{1}{4}\int\limits_0^1 dt (y\log y - y)\Big|_{y=0}^{X^2(t)}$$

$$= \frac{X^2}{4}\int\limits_0^1 \frac{dt}{t^2+1}\left(\log\frac{X^2}{t^2+1} - 1\right) = \frac{\pi}{16}X^2(2\log X - 1) - \frac{X^2}{4}I_0.$$

Replace I_0 in the latter formula by its value, which was found above, and obtain the required relation.

Theorem 2. *Let $1 \le U \le R$, $R_1 = R/U$. Then the number T_2 of solutions of system* (3) *with the additional restriction $Q' > U$ satisfies the asymptotic formula*

$$T_2 = \frac{\pi}{4}\cdot\frac{R^2}{\zeta^2(2)}\left[\log(x+1)\log R_1 + C_2(x)\right] + O(R^2U^{-1/4}\log^2 R) + O(RU^2\log^2 R),$$

in which

$$C_2(x) = \log(x+1)\left(2\frac{C}{\pi} - \frac{\zeta'(2)}{\zeta(2)} + \gamma - \frac{1}{2}\log(x+1) + \log\frac{x}{2} - \frac{3}{2}\right) + h(x) \tag{22}$$

and $h(x)$ is the function defined by relation (20).

Proof. Apply Corollary 2 and take into account Lemmas 10 and 11.

6. The main result

Theorem 3. *Let $R \geq 2$. Then the number $N_x(R)$ satisfies the asymptotic formula*

$$N_x(R) = \frac{3}{\pi} R^2 \left[\log(x+1)\log R + C(x)\right] + O(R^{17/9}\log^2 R),$$

in which

$$C(x) = C_1(x) + C_2(x) + \log(x+1)\frac{\zeta'(2)}{\zeta(2)} + \zeta(2)\arctan x(1 - 2[x=1])$$

and $C_1(x)$ and $C_2(x)$ are functions that are defined by relations (18) *and* (22).

Proof. Theorems 1 and 2 imply the relation

$$T_x^*(R) = T_1 + T_2 = \frac{\pi}{4}\cdot\frac{R^2}{\zeta^2(2)}\left[\log(x+1)\log R + C_1(x) + C_2(x)\right]$$

$$+O(R^2U^{-1/2}\log^5 R) + O(R^2U^{-1/4}\log^2 R) + O(RU^2\log^2 R).$$

Choose $U = R^{4/9}$ and substitute the result into formula (5); we obtain

$$N_x^*(R) = \frac{\pi}{2}\cdot\frac{R^2}{\zeta^2(2)}\left[\log(x+1)\log R + C_3(x)\right] + O(R^{17/9}\log^2 R),$$

where

$$C_3(x) = C_1(x) + C_2(x) + \zeta(2)\arctan x(1 - 2[x=1]).$$

Finally, we apply the relation

$$N_x(R) = \sum_{d \leq R} N_x^*(R/d)$$

and complete the proof of the theorem.

Remark. As a result, we arrive at the following representation of the constant $C(x)$:

$$C(x) = \log(x+1)\left(2\frac{C}{\pi} + \gamma - \frac{\log(x+1)}{2} + \log\frac{x}{2} - \frac{3}{2}\right)$$

$$+h(x) + \frac{2}{\pi}\left(f(x) + g(x) - \arctan\frac{x}{2+3x}\right),$$

where $f(x)$, $g(x)$, and $h(x)$ are functions defined by relations (14), (17), (20); most likely, these functions cannot be evaluated explicitly.

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