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A Discrete Analog of the Poisson Summation Formula

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Abstract—The first part of this paper is concerned with the proof of a discrete analog of the Poisson summation formula. In the second part, we describe an elementary proof of a functional equation for the function $\theta(t)$, based on the summation formula derived in the paper.

Key words: *Poisson summation formula, Gauss sum, uniform grid, Fourier series.*

1. INTRODUCTION

Suppose that $\mathcal{S}$ is the space of infinitely differentiable functions $f\colon \mathbb{R} \to \mathbb{C}$, decreasing faster than any positive power, i.e., for any positive integer n,

$$\lim_{x\to\pm\infty} |x|^n f(x) = 0.$$

We define the Fourier transform $\widehat{f}$ of a function $f \in \mathcal{S}$ by the formula

$$\widehat{f}(y) = \int_{-\infty}^{\infty} e^{2\pi ixy} f(x)\,dx.$$

Such an integral is convergent for all real values of y and determines the function $\widehat{f}(y) \in \mathcal{S}$.

It is well known that the sum of the values of a function at points of a uniform grid is related to a similar sum of the values of its Fourier transform. A similar relationship is described by the Poisson summation formula. There are different versions of this formula. For a function $f \in \mathcal{S}$, it can be written without a remainder [1]:

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} \widehat{f}(m). \tag{1}$$

The Poisson summation formula is used in various problems of mathematical analysis and number theory. For example, using formula (1), we can prove that the function

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2} \tag{2}$$

defined for $t > 0$ satisfies the functional equation

$$\theta(t^{-1}) = \sqrt{t}\,\theta(t) \tag{3}$$

(see [1, 2]). Different versions of the Poisson summation formula with remainder allow us to find an exact value of the Gauss sum (see [3, 2]) and to obtain estimates of trigonometric sums (see [2]). A few other examples can be found in the book [4].

In the present paper, we consider functions defined at a finite number of points of a uniform grid. For such functions, we can prove a discrete analog of formula (1), which relates the sum of the values of a function at the nodes of a spaced-out uniform grid to the sum of its finite Fourier coefficients. Next, the resulting formula is applied to the proof of the functional equation for the function $\theta(t)$. This approach to the proof of relation (3) is elementary.

Similarly, a discrete analog of relation (1) can also be used in other problems to which the ordinary Poisson summation formula is applied.

2. A DISCRETE ANALOG OF THE POISSON SUMMATION FORMULA

In what follows, we consider a uniform grid consisting of points with integer coordinates, i.e., we assume that the function $f(x)$ is defined for all integer values of x in the interval $0 \le x < p$, where p is a positive integer.

It is well known that at each of these points a function $f(x)$ can be expressed by its finite Fourier series

$$f(x) = \sum_{k=0}^{p-1} C_p(k) e^{2\pi i k x/p}, \qquad 0 \le x < p, \tag{4}$$

where the $C_p(k)$ are the finite Fourier coefficients of the function $f(x)$ and can be determined by the formula

$$C_p(k) = \frac{1}{p} \sum_{x=0}^{p-1} f(x) e^{-2\pi i k x/p}, \qquad 0 \le k < p.$$

The following assertion can be considered a discrete analog of formula (1).

Theorem 1. *Suppose that p_1, p_2 are positive integers, $p = p_1 p_2$, the function $f(x)$ is defined for all integer values of x in the interval $0 \le x < p$, and the $C_p(k)$ are the finite Fourier coefficients of $f(x)$. Then the following relation is valid*:

$$\sum_{y=0}^{p_2-1} f(p_1 y) = p_2 \sum_{n=0}^{p_1-1} C_p(p_2 n). \tag{5}$$

Proof. Let us transform the first sum using formula (4):

$$\sum_{y=0}^{p_2-1} f(p_1 y) = \sum_{y=0}^{p_2-1} \sum_{k=0}^{p-1} C_p(k) e^{2\pi i k p_1 y/p} = \sum_{k=0}^{p-1} C_p(k) \sum_{y=0}^{p_2-1} e^{2\pi i k y/p_2}.$$

Further, since

$$\sum_{y=0}^{p_2-1} e^{2\pi i k y/p_2} = p_2 \delta_{p_2}(k) = \begin{cases} p_2 & \text{if } k \equiv 0 \pmod{p_2}, \\ 0 & \text{if } k \not\equiv 0 \pmod{p_2}, \end{cases}$$

we have

$$\sum_{y=0}^{p_2-1} f(p_1 y) = \sum_{k=0}^{p-1} C_p(k) p_2 \delta_{p_2}(k) = p_2 \sum_{n=0}^{p_1-1} C_p(p_2 n). \quad \square$$

3. PROOF OF THE FUNCTIONAL EQUATION FOR THE FUNCTION $\theta(t)$

Before proving relation (3), we consider several auxiliary assertions.

Lemma 1. *Suppose that q_1, q_2 are positive integers and $q = 2q_1q_2$. Then the following relation is valid:*

$$\frac{q_1}{2^q}\sum_{m=-q_2}^{q_2}\binom{q}{q_1(q_2+m)} = \sum_{n=0}^{q_1-1}\left(\cos\frac{\pi n}{q_1}\right)^q. \tag{6}$$

Proof. Consider the function $f(x) = \binom{q}{x}$ defined for integer values of x in the interval $0 \le x < q$ and find its finite Fourier coefficients:

$$\begin{aligned}C_q(k) &= \frac{1}{q}\sum_{x=0}^{q-1}\binom{q}{x}e^{-2\pi ikx/q} = \frac{1}{q}[(1+e^{-2\pi ik/q})^q - 1]\\ &= \frac{1}{q}[e^{-\pi ik}(e^{\pi ik/q}+e^{-\pi ik/q})^q - 1] = \frac{1}{q}\left[(-1)^k\left(2\cos\frac{\pi k}{q}\right)^q - 1\right].\end{aligned}$$

Applying Theorem 1 to the function $f(x) = \binom{q}{x}$ with $p_1 = q_1$, $p_2 = 2q_2$, and $p = 2q_1q_2$, we obtain the relation

$$\sum_{y=0}^{2q_2-1}\binom{q}{q_1y} = \frac{2q_2}{q}\sum_{n=0}^{q_1-1}\left(2\cos\frac{\pi n}{q_1}\right)^q - 1.$$

Hence

$$\frac{q_1}{2^q}\sum_{y=0}^{2q_2}\binom{q}{q_1y} = \sum_{n=0}^{q_1-1}\left(\cos\frac{\pi n}{q_1}\right)^q.$$

This yields the assertion of the lemma. □

Lemma 2. *Suppose that $t > 0$ is a real number such that the product πt is rational. Further, let*

$$\pi t = \frac{a}{b}, \quad (a,b) = 1, \quad z \ge 1, \quad q_1 = az, \quad q_2 = bz, \quad q = 2q_1q_2.$$

We shall also assume that the numbers M, N, m, and n satisfy the inequalities

$$0 \le M, N \le \sqrt{z}, \qquad |m| \le M, \qquad |n| \le N.$$

Then the following asymptotic formulas are valid:

$$\frac{q_1}{2^q}\binom{q}{q_1(q_2+m)} = \sqrt{t}e^{-\pi tm^2}\left(1 + O\left(\frac{M^4}{z^2}\right)\right), \tag{7}$$

$$\left(\cos\frac{\pi n}{q_1}\right)^q = e^{-\pi n^2/t}\left(1 + O\left(\frac{N^4}{z^2}\right)\right), \tag{8}$$

where the constants under the sign of O may depend on a and b.

Proof. First, let us verify relation (7). We use Stirling's formula

$$k! = \sqrt{2\pi k}\left(\frac{k}{e}\right)^k\left(1 + O\left(\frac{1}{k}\right)\right)$$

to calculate the binomial coefficient $\binom{q}{q_1(q_2+m)}$:

$$\binom{q}{q_1(q_2+m)} = \frac{(2q_1q_2)!}{[q_1(q_2+m)]!\,[q_1(q_2-m)]!}$$
$$= \sqrt{\frac{4\pi q_1q_2}{4\pi^2q_1^2(q_2^2-m^2)}}\cdot\frac{(2q_2)^{2q_1q_2}(1+O(z^{-2}))}{(q_2+m)^{q_1(q_2+m)}(q_2-m)^{q_1(q_2-m)}}.$$

Hence

$$\frac{q_1}{2^q}\binom{q}{q_1(q_2+m)} = \sqrt{t}\biggl(1+\frac{m}{q_2}\biggr)^{-q_1(q_2+m)}\biggl(1-\frac{m}{q_2}\biggr)^{-q_1(q_2-m)}\biggl(1+O\biggl(\frac{M^2}{z^2}\biggr)\biggr).$$

Further, noting that

$$\biggl(1+\frac{m}{q_2}\biggr)^{q_1(q_2+m)}\biggl(1-\frac{m}{q_2}\biggr)^{q_1(q_2-m)}$$
$$= \exp\biggl(q_1q_2\ln\biggl(1-\frac{m^2}{q_2^2}\biggr)+q_1m\ln\biggl(1+\frac{m}{q_2}\biggr)-q_1m\ln\biggl(1-\frac{m}{q_2}\biggr)\biggr)$$
$$= \exp\biggl(\frac{q_1}{q_2}m^2+O\biggl(\frac{M^4}{z^2}\biggr)\biggr) = e^{\pi tm^2}\biggl(1+O\biggl(\frac{M^4}{z^2}\biggr)\biggr),$$

we obtain relation (7).

The proof of formula (8) is carried out in a similar way:

$$\biggl(\cos\frac{\pi n}{q_1}\biggr)^q = \exp\biggl(q\ln\biggl(1-\frac{\pi^2n^2}{2q_1^2}+O\biggl(\frac{N^4}{z^4}\biggr)\biggr)\biggr) = \exp\biggl(q\biggl(-\frac{\pi^2n^2}{2q_1^2}+O\biggl(\frac{N^4}{z^4}\biggr)\biggr)\biggr)$$
$$= \exp\biggl(-\frac{q_2\pi^2n^2}{q_1}+O\biggl(\frac{N^4}{z^2}\biggr)\biggr) = e^{-\pi n^2/t}\biggl(1+O\biggl(\frac{N^4}{z^2}\biggr)\biggr). \qquad \square$$

Lemma 3. *Suppose that, just as in Lemma 2, the following conditions are satisfied*:

$$t>0,\qquad \pi t=\frac{a}{b},\qquad (a,b)=1,\qquad z\ge 1,\qquad q_1=az,\qquad q_2=bz,\qquad q=2q_1q_2.$$

We also assume that the parameters M and N satisfy the inequalities

$$\frac{1}{t}\le M\le\sqrt{z},\qquad t\le N\le\sqrt{z}.$$

Then the following estimates are valid:

$$\sum_{m\ge M} e^{-\pi tm^2} = O(e^{-2M}), \tag{9}$$

$$\sum_{n\ge N} e^{-\pi n^2/t} = O(e^{-2N}), \tag{10}$$

$$\frac{q_1}{2^q}\sum_{m\ge M}\binom{q}{q_1(q_2+m)} = O(e^{-2M}), \tag{11}$$

$$\sum_{N\le n\le q_1/2}\biggl(\cos\frac{\pi n}{q_1}\biggr)^q = O(e^{-2N}), \tag{12}$$

where, as above, the constants under the sign of O may depend on a and b.

Proof. Let us verify that in each of the four sums the summands decrease no slower than the elements of a geometric progression with the common ratio $1/2$. Hence each of the sums can be estimated by the first (largest) summand.

Indeed, in the first case

$$\frac{e^{-\pi t(m+1)^2}}{e^{-\pi t m^2}} = e^{-\pi t(2m+1)} < e^{-2\pi t M} \le e^{-2\pi} < \frac{1}{2}$$

and the estimate for the largest summand is

$$e^{-\pi t M^2} \le e^{-\pi M} = O(e^{-2M}).$$

The estimate (10) can be verified in exactly the same way.

Let us prove formula (11). The ratio of adjacent summands is again at most $1/2$:

$$\frac{\binom{q}{q_1(q_2+m+1)}}{\binom{q}{q_1(q_2+m)}} = \frac{[q_1(q_2-m)]\cdots[q_1(q_2-m)-q_1+1]}{[q_1(q_2+m)+q_1]\cdots[q_1(q_2+m)+1]} < \left(\frac{q_2-m}{q_2+m}\right)^q$$
$$< \left(1+\frac{m}{q_2}\right)^{-2q_1} = e^{-2q_1\ln(1+m/q_2)} < e^{-mq_1/q_2} = e^{-\pi t m} < e^{-\pi} < \frac{1}{2}.$$

In addition, the first summand on the left-hand side of (11) can be estimated using Lemma 2:

$$\frac{q_1}{2^q}\binom{q}{q_1(q_2+M)} = O(e^{-\pi t M^2}) = O(e^{-2M}).$$

To verify formula (12), first note that

$$\frac{\cos(\pi(n+1)/q_1)}{\cos(\pi n/q_1)} = 1 + \frac{\cos(\pi(n+1)/q_1) - \cos(\pi n/q_1)}{\cos(\pi n/q_1)}$$
$$< 1 - \frac{(\pi/q_1)\sin(\pi n/q_1)}{\cos(\pi n/q_1)} = 1 - \frac{\pi}{q_1}\tan\frac{\pi n}{q_1} < 1 - \frac{\pi^2 n}{q_1^2}.$$

Therefore,

$$\left(\frac{\cos(\pi(n+1)/q_1)}{\cos(\pi n/q_1)}\right)^q < \left(1-\frac{\pi^2 n}{q_1^2}\right)^q < e^{-2q_2\pi^2 n/q_1} = e^{-\pi n/t} < e^{-\pi} < \frac{1}{2}.$$

The first summand on the left-hand side of (12) can again be estimated using Lemma 2:

$$\left(\cos\frac{\pi N}{q_1}\right)^q = O(e^{-\pi N^2/t}) = O(e^{-2N}). \quad \square$$

Theorem 2. *For all $t>0$, the function $\theta(t)$ defined by the series* (4) *satisfies relation* (3).

Proof. The absolute convergence of the series (4) implies the continuity of the function $\theta(t)$. Therefore, it suffices to prove the theorem only for positive t for which the number πt is rational.

We choose $t>0$ and define the integers a and b by the relation $\pi t = a/b$, $(a,b)=1$. Further, choose $z \ge e^{\max(t,t^{-1})}$ and set

$$M = N = \ln z, \qquad q_1 = az, \qquad q_2 = bz, \qquad q = 2q_1q_2.$$

Using formulas (11) and (7) successively, we find

$$\begin{aligned}\frac{q_1}{2^q}\sum_{m=-q_2}^{q_2}\binom{q}{q_1(q_2+m)} &= \frac{q_1}{2^q}\sum_{|m|<M}\binom{q}{q_1(q_2+m)} + O\left(\frac{1}{z^2}\right)\\ &= \sqrt{t}\left(1+O\left(\frac{M^4}{z^2}\right)\right)\sum_{|m|<M} e^{-\pi t m^2} + O\left(\frac{1}{z^2}\right)\\ &= \sqrt{t}\sum_{|m|<M} e^{-\pi t m^2} + O\left(\frac{M^4}{z^2}\right).\end{aligned}$$

Applying the estimate (9), we obtain the relation

$$\frac{q_1}{2^q}\sum_{m=-q_2}^{q_2}\binom{q}{q_1(q_2+m)} = \sqrt{t}\theta(t) + O\left(\frac{M^4}{z^2}\right). \tag{13}$$

Similarly, it follows from formulas (12) and (8) that

$$\begin{aligned}\sum_{n=0}^{q_1-1}\left(\cos\frac{\pi n}{q_1}\right)^q &= \sum_{|n|\le N}\left(\cos\frac{\pi n}{q_1}\right)^q + O\left(\frac{1}{z^2}\right) = \left(1+O\left(\frac{N^4}{z^2}\right)\right)\sum_{|n|<N} e^{-\pi n^2/t} + O\left(\frac{1}{z^2}\right)\\ &= \sum_{|n|<N} e^{-\pi n^2/t} + O\left(\frac{N^4}{z^2}\right).\end{aligned}$$

Now, from the estimate (10) we obtain the relation

$$\sum_{n=0}^{q_1-1}\left(\cos\frac{\pi n}{q_1}\right)^q = \theta\left(\frac{1}{t}\right) + O\left(\frac{N^4}{z^2}\right). \tag{14}$$

Substituting formulas (13) and (14) into (6), we find

$$\sqrt{t}\,\theta(t) = \theta\left(\frac{1}{t}\right) + O\left(\frac{\ln^4 z}{z^2}\right).$$

Passing to the limit as $z \to \infty$, we obtain the required result. □

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