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A Discrete Analog of Euler's Summation Formula

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Abstract—In this paper, we prove a discrete analog of Euler's summation formula. The difference from the classical Euler formula is in that the derivatives are replaced by finite differences and the integrals by finite sums. Instead of Bernoulli numbers and Bernoulli polynomials, special numbers P_n and special polynomials $P_n(x)$ introduced by Korobov in 1996 appear in the formula.

Key words: *Euler's summation formula, Bernoulli polynomials, Fourier series.*

1. INTRODUCTION

In [1] Korobov introduced special numbers P_n and special polynomials $P_n(x)$, which can be called *discrete analogs* of Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$. The analogy consists in that the properties of the polynomials $B_n(x)$ are transformed into those of the polynomials $P_n(x)$ under the replacement of continuous entities by discrete ones: derivatives by finite differences, integrals by finite sums, and Fourier series by finite Fourier series.

One of the important applications of the Bernoulli polynomials is Euler's summation formula, which establishes the connection between the sum of the values of a sufficiently smooth function $f(x)$ at the points of a uniform grid

$$\sum_{k=a}^{b-1} f(k) \tag{1}$$

and the definite integral of this function over the corresponding closed interval:

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx + \sum_{\nu=1}^{n} \frac{B_\nu}{\nu!} f^{(\nu-1)}(x)\Big|_a^b - R_n[f], \tag{2}$$

where

$$R_n[f] = \int_a^b \frac{B_n(1-\{x\})}{n!} f^{(n)}(x)\,dx = (-1)^n \int_a^b \frac{B_n(\{x\})}{n!} f^{(n)}(x)\,dx.$$

Another application of the Bernoulli polynomials is the following formula for the representation of an arbitrary function $f(x)$ on the closed interval $[0\,;\,1]$:

$$f(x) = \int_0^1 f(y)\,dy + \sum_{k=1}^{n} \frac{B_k(x)}{k!}\big(f^{(k-1)}(y)\big|_0^1\big) - R_n, \tag{3}$$

where

$$R_n = \int_0^1 \frac{B_n(\{x-y\})}{n!} f^{(n)}(y)\,dy = (-1)^n \int_0^1 \frac{B_n(1-\{x-y\})}{n!} f^{(n)}(y)\,dy$$

(see, for example, [2]).

The shortcoming of using formulas (2) and (3) for solving applied problems is that one must know the derivatives of the function $f(x)$.

In the present paper, we prove discrete analogs of formulas (2) and (3) (Theorems 2 and 1). In the discrete analog of Euler's formula, the sum (1) is not approximated by an integral, but by the sum of the values of the same function at the nodes of a denser uniform grid:

$$\frac{1}{p}\sum_{y=ap}^{bp-1} f\left(\frac{y}{p}\right). \tag{4}$$

In the problem of finding the sum (1), such a formula turns out to be more convenient if the sum (4) can be calculated more precisely than the integral

$$\int_a^b f(x)\,dx. \tag{5}$$

On the other hand, the discrete analog of Euler's summation formula is more convenient to use in problems of approximate analysis, since this formula connects the sum (4), close to the integral (5), with the shorter sum (1). Moreover, in contrast to the ordinary Euler summation formula, the application of its discrete analog does not involve the values of the derivatives of the function, but only the values of the function at a finite number of points.

2. DISCRETE ANALOG OF EULER'S SUMMATION FORMULA

Suppose that $n \ge 0$, $p \ge 2$ are integers. For any real x, the number P_n and the polynomials $P_n(x)$ are defined by the relations

$$P_0 = 1, \qquad C_p^1 P_n + \dots + C_p^{n+1} P_0 = 0 \quad (n \ge 1);$$
$$P_0(x) = 1, \qquad P_n(x) = P_0 C_x^n + \dots + P_{n-1} C_x^1 + P_n \quad (n \ge 1).$$

Using the definition, it is easy to calculate the first special numbers and special polynomials

$$P_1 = -\frac{p-1}{2}, \qquad P_2 = \frac{p^2-1}{12}, \qquad P_3 = -\frac{p^2-1}{24},$$
$$P_1(x) = x - \frac{p-1}{2}, \qquad P_2(x) = \frac{1}{2}x^2 - \frac{p}{2}x + \frac{p^2-1}{12},$$
$$P_3(x) = \frac{1}{6}x^3 - \frac{p+1}{4}x^2 + \frac{p^2+3p}{12}x - \frac{p^2-1}{24}.$$

Let us define the action of the finite difference operator Δ by the relations

$$\Delta^0 f\left(\frac{x}{p}\right) = f\left(\frac{x}{p}\right), \qquad \Delta^1 f\left(\frac{x}{p}\right) = f\left(\frac{x+1}{p}\right) - f\left(\frac{x}{p}\right).$$

The finite difference of order $n \ge 1$ is defined by the recurrence relation

$$\Delta^n f\left(\frac{x}{p}\right) = \Delta^1\left(\Delta^{n-1} f\left(\frac{x}{p}\right)\right).$$

Note two properties of the special polynomials proved in [1]:

1° $P_n(0) = P_n \quad (n \ge 0)$;

2° for an integer $x \in [0\,;\, p+n-2]$,

$$P_n(x) = -\sum_{m=1}^{p-1} \frac{e^{2\pi i m/p}}{(e^{2\pi i m/p}-1)^n} e^{2\pi i m x/p} \qquad (n \ge 0).$$

First, let us prove the discrete analog of formula (3).

Theorem 1. *Suppose that n, k are integers, $n \geq 1$, and the function $f(z/p)$ is defined at all points of the interval $k \leq z/p < k+1+n/p$. Then, for all integers x from the interval $0 \leq x \leq p$, the following relation is valid*:

$$f\left(k+\frac{x}{p}\right)=\frac{1}{p}\sum_{y=0}^{p-1} f\left(k+\frac{y}{p}\right)+\frac{1}{p}\sum_{\nu=1}^{n} P_\nu(x)\Delta^{\nu-1} f\left(k+\frac{y}{p}\right)\Big|_{y=0}^{p}-R_n, \tag{6}$$

where the remainder R_n can be written in two forms:

$$R_n=\frac{1}{p}\sum_{y=0}^{p-1}\Delta^n f\left(k+\frac{y}{p}\right)P_n\left(p\left\{\frac{p+x-y-1}{p}\right\}\right), \tag{7}$$

$$R_n=\frac{(-1)^n}{p}\sum_{y=0}^{p-1}\Delta^n f\left(k+\frac{y}{p}\right)P_n\left(p\left\{\frac{y-x-n-1}{p}\right\}\right). \tag{8}$$

Proof. First, let us prove formulas (6) with remainder (7). For $n=1$, it is necessary to verify the relation

$$f\left(k+\frac{x}{p}\right)=\frac{1}{p}\sum_{y=0}^{p-1} f\left(k+\frac{y}{p}\right)+\frac{1}{p}\left(x-\frac{p-1}{2}\right)[f(k+1)-f(k)]-R_1. \tag{9}$$

To do this, let us transform the remainder R_1:

$$\begin{aligned}
R_1&=\frac{1}{p}\sum_{y=0}^{p-1}\Delta f\left(k+\frac{y}{p}\right)P_1\left(p\left\{\frac{p+x-y-1}{p}\right\}\right)\\
&=\frac{1}{p}\sum_{y=0}^{x-1}\Delta f\left(k+\frac{y}{p}\right)\left(x-y-1-\frac{p-1}{2}\right)+\frac{1}{p}\sum_{y=x}^{p-1}\Delta f\left(k+\frac{y}{p}\right)\left(p+x-y-1-\frac{p-1}{2}\right)\\
&=f(k+1)-f\left(k+\frac{x}{p}\right)+\frac{1}{p}\sum_{y=0}^{x-1}\Delta f\left(k+\frac{y}{p}\right)\left(x-y-1-\frac{p-1}{2}\right).
\end{aligned}$$

Let us apply the Abel transformation to the last sum

$$\sum_{y=0}^{p-1}F(y)\varphi(y)=F(p)\sum_{y=0}^{p-1}\varphi(y)-\sum_{y=0}^{p-1}\big(F(y+1)-F(y)\big)\sum_{z=0}^{y}\varphi(z), \tag{10}$$

setting $\varphi(y)=\Delta f(k+y/p)$ and $F(y)=x-y-1-(p-1)/2$:

$$\begin{aligned}
R_1+f\left(k+\frac{x}{p}\right)&=f(k+1)+\frac{1}{p}\left(x-p-1-\frac{p-1}{2}\right)[f(k+1)-f(k)]\\
&\qquad+\frac{1}{p}\sum_{y=0}^{p-1}\left[f\left(k+\frac{y+1}{p}\right)-f(k)\right]\\
&=\frac{1}{p}\sum_{y=0}^{p-1}f\left(k+\frac{y}{p}\right)+[f(k+1)-f(k)]\left(x-\frac{p-1}{2}\right).
\end{aligned}$$

Thus relation (9) is proved.

Further, let us apply the induction method. Suppose that relations (6), (7) are valid for some $n \geq 1$.

Property 2° implies the following relation for all integers x:

$$P_n\left(p\left\{\frac{x}{p}\right\}\right) = -\sum_{m=1}^{p-1} \frac{e^{2\pi im/p}}{(e^{2\pi im/p}-1)^n} e^{2\pi imx/p} \qquad (n \geq 0). \tag{11}$$

Summing this expression, we obtain the relations

$$\sum_{y=0}^{p-1} P_n\left(p\left\{\frac{p+x-y-1}{p}\right\}\right) = 0, \tag{12}$$

$$\sum_{z=0}^{y} P_n\left(p\left\{\frac{x+x-z-1}{p}\right\}\right) = P_{n+1}\left(p\left\{\frac{x}{p}\right\}\right) - P_{n+1}\left(p\left\{\frac{p+x-y-1}{p}\right\}\right). \tag{13}$$

Let us apply the Abel transformation (10) again to the remainder R_n, choosing

$$F(y) = \Delta^n f\left(k + \frac{y}{p}\right) \qquad \text{and} \qquad \varphi(y) = P_n\left(p\left\{\frac{p+x-y-1}{p}\right\}\right).$$

In view of formulas (12) and (13), we obtain the required relation

$$R_n = -\frac{1}{p} P_{n+1}(x) \Delta^n f\left(k + \frac{y}{p}\right)\bigg|_0^p + R_{n+1},$$

which proves formula (6) with remainder in the first form (7).

To prove the second representation of the remainder (8), it suffices to use the relation

$$P_n\left(p\left\{\frac{p-1-x}{p}\right\}\right) = (-1)^n P_n\left(p\left\{\frac{x+n-1}{p}\right\}\right) \qquad (n \geq 0), \tag{14}$$

which is a direct consequence of formula (11). The theorem is proved. □

Let us now prove the discrete analog of Euler's summation formula (2).

Theorem 2. *Suppose that $b > a$ and $n \geq 1$ are integers and the function $f(z/p)$ is defined at all points of the interval $a \leq z/p < b + n/p$. Then the following relation is valid*:

$$\sum_{k=a}^{b-1} f(k) = \frac{1}{p} \sum_{y=ap}^{bp-1} f\left(\frac{y}{p}\right) + \frac{1}{p} \sum_{\nu=1}^{n} P_\nu \Delta^{\nu-1} f\left(\frac{y}{p}\right)\bigg|_{ap}^{bp} - R_n[f], \tag{15}$$

where the remainder $R_n[f]$ can be written in two forms:

$$R_n[f] = \frac{1}{p} \sum_{y=ap}^{bp-1} \Delta^n f\left(\frac{y}{p}\right) P_n\left(p\left\{\frac{p-y-1}{p}\right\}\right), \tag{16}$$

$$R_n[f] = \frac{(-1)^n}{p} \sum_{y=ap}^{bp-1} \Delta^n f\left(\frac{y}{p}\right) P_n\left(p\left\{\frac{y+n-1}{p}\right\}\right). \tag{17}$$

Proof. If in relation (6) we set $x = 0$ and sum it over k from a to $b-1$, then we obtain formula (15) with remainder $R_n[f]$ in the form (16). The second form of the remainder (17), just as in Theorem 1, is a consequence of formula (14). The theorem is proved. □

Remark. There exist different estimates of the remainder in Euler's summation formula (2); they depend on the behavior of the derivatives of the function $f(x)$ (see [3]). These estimates are based on the geometric properties of Bernoulli polynomials. For example, the following inequality is always valid:

$$|R_{2n-1}| \le \frac{(b-a)|B_{2n}|}{(2n)!} \max_{\xi\in[a\,;\,b]} |f^{(2n)}(\xi)|.$$

By analogy, we can prove that the remainder in formula (15) satisfies the estimate

$$|R_{2n-1}[f]| \le 2(b-a)p^{2n}\left(\frac{|B_{2n}|}{(2n)!} + \frac{1}{p^2}\right) \max_{y\in[ap\,;\,bp-1]} \left|\Delta^{2n} f\left(\frac{y}{p}\right)\right|.$$

The author plans to study the geometric properties of special polynomials and different estimates of the remainder in the summation formula (15) in a special paper.

3. SYMBOLIC DERIVATION OF THE DISCRETE ANALOG OF EULER'S SUMMATION FORMULA

It is well known that Euler's formula can be obtained by means of a nonstrict symbolic derivation based on the formal connection between the shift, differentiation, and finite difference operators (see, for example, [3]). Its discrete analog can also be obtained using formal arguments. Of course, the calculations given below cannot be regarded as a proof. They are only an argument in support of the existence of such a formula. For the symbolic derivation of Euler's formula, we use the generating function of the sequence of special numbers. As was noted in [1], this function is of the form

$$\sum_{n=0}^{\infty} P_n t^n = \frac{pt}{(t+1)^p - 1}.$$

By E we denote the shift operator, whose action is defined by the relation

$$Ef\left(\frac{y}{p}\right) = f\left(\frac{y+1}{p}\right).$$

Let $Q = b-a$. The left-hand side of (15) can be written as

$$\sum_{k=a}^{b-1} f(k) = (1 + E^p + E^{2p} + \dots + E^{(Q-1)p}) f\left(\frac{ap}{p}\right) = \frac{E^{Qp}-1}{E^p - 1} f\left(\frac{ap}{p}\right).$$

The sum of the values of the function at the rational points with denominator p takes the form

$$\frac{1}{p}\sum_{y=ap}^{bp-1} f\left(\frac{y}{p}\right) = \frac{1}{p}(1 + E + E^2 + \dots + E^{Qp-1}) f\left(\frac{ap}{j}\right) = \frac{1}{p}\cdot\frac{E^{Qp}-1}{E-1} f\left(\frac{ap}{p}\right).$$

If we use the operator relation $E - 1 = \Delta$, then the difference of the operators applied in the first and second cases can be expressed symbolically as

$$\frac{E^{Qp}-1}{p}\left(\frac{p}{E^p-1} - \frac{1}{E-1}\right) = \frac{E^{Qp}-1}{p\Delta}\left(\frac{p\Delta}{(\Delta+1)^p - 1} - 1\right).$$

On the right-hand side, we have the generating function of the special numbers with argument Δ. Expanding it in powers of Δ, we obtain the relation

$$\frac{E^{Qp}-1}{p}\left(\frac{p}{E^p-1}-\frac{1}{E-1}\right)=\frac{E^{Qp}-1}{p}\sum_{\nu=1}^{\infty}P_\nu\Delta^{\nu-1}.$$

Applying the resulting operator relation to $f(ap/p)$, we obtain the discrete analog of Euler's summation formula without remainder:

$$\sum_{k=a}^{b-1}f(k)-\frac{1}{p}\sum_{y=ap}^{bp-1}f\left(\frac{y}{p}\right)=\frac{1}{p}\sum_{\nu=1}^{\infty}P_\nu\Delta^{\nu-1}f\left(\frac{y}{p}\right)\bigg|_{ap}^{bp}.$$

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