Mathematical Notes, Vol. 64, No. 2, 98

On the Number of Summands in the Asymptotic Formula for the Number of Solutions to Waring's Equation

A. V. Ustinov

UDC 511.3

Abstract. In the paper, an estimate of the number of summands in the asymptotic formula for the number of solutions to Waring's equation is obtained. This is achieved by means of a recurrent process leading to a greater reduction than that in Vinogradov's mean value theorem.

Key words: Waring's , Waring's equation, estimate of the number of summands, Fourier s, Hardy's , Vinogradov's mean value theorem.

§1. Introduction

Let $N_k^{(P)}(\lambda_1,\dots,\lambda_n)$ be the number of solutions of the system

$$\begin{cases} x_1+\dots+x_k-y_1-\dots-y_k=\lambda_1,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_1^n+\dots+x_k^n-y_1^n-\dots-y_k^n=\lambda_n,\end{cases}\qquad 0\le x_1,\dots,y_k<P.$$

(Here and subsequently the variables take only integer values.) By Vinogradov's mean value theorem, for $k\ge n\tau$ the estimate holds:

$$N_k(P)=N_k^{(P)}(0,\dots,0)\ll P^{2k-\frac{n(n+1)}{2}+\frac{n^2}{2}\left(1-\frac1n\right)^\tau}. \tag{1}$$

Denote by $I_k(P)$ the number of solutions of Hardy's equation

$$x_1^n+\dots+x_k^n-y_1^n-\dots-y_k^n=0,\qquad 0\le x_1,\dots,y_k<P.$$

Formula (1) allows us to obtain the estimate for $I_k(P)$:

$$I_k(P)\ll\sum_{|\lambda_1|<kP}\cdots\sum_{|\lambda_{n-1}|<kP^{n-1}}N_k^{(P)}(\lambda_1,\dots,\lambda_{n-1},0)\ll P^{n(n-1)/2}N_k(P)\ll P^{2k-n+\frac{n^2}{2}\left(1-\frac1n\right)^\tau}. \tag{2}$$

Let $I(N)$ be the number of solutions to Waring's equation

$$x_1^n+\dots+x_k^n=N,\qquad 0\le x_1,\dots,x_k\le N^{1/n}.$$

In [1] it was proved that for $n\ge4$ and $k\ge2[n^2(2\ln n+\ln\ln n+5)]$ the asymptotic formula is valid:

$$I(N)=\sigma(N)\gamma N^{k/n-1}+O(N^{k/n-1-1/(20n\ln n)}), \tag{3}$$

where $\gamma=\big(\Gamma(1+1/n)\big)^k/\Gamma(k/n)$, $\sigma(N)\ge c_0(n,k)>0$.

In the present paper we prove that for $k\ge n(n-1)+n\tau$ the estimate holds:

$$I_k(P)\ll P^{2k-n+\frac n2\left(1-\frac1n\right)^\tau}.$$

This result refines the estimate (2); this helps prove that formula (3) is valid for

$$n\ge4\qquad\text{and}\qquad k\ge2[n^2(\ln n+\ln\ln n+6)].$$

Translated from *Matematicheskie Zametki*, Vol. 64, No. 2, pp. 285–296, August, 98.
Original article submitted March 23, 1998.

 0001–4346/98/6412–0240 $20.00

§2. Properties of the Fourier coefficients of some functions

In [2] the following assertion was proved.

LEMMA 1. *Suppose $N_1, \dots, N_n$ are nonnegative integers and $F(\alpha_1, \dots, \alpha_n)$ is a nonnegative real function defined on the cube $E_n = [0,1]^n$ and Lebesgue integrable. Suppose that the Fourier coefficients $c(\lambda_1, \dots, \lambda_n)$ of $F(\alpha_1, \dots, \alpha_n)$ are also nonnegative real numbers. Then, for any integers $\mu_1, \dots, \mu_n$, the is valid*:

$$\sum_{|\lambda_1|\le N_1} \cdots \sum_{|\lambda_n|\le N_n} c(\lambda_1+\mu_1, \dots, \lambda_n+\mu_n) \le 4^n \sum_{|\lambda_1|\le N_1} \cdots \sum_{|\lambda_n|\le N_n} c(\lambda_1, \dots, \lambda_n).$$

In subsequent arguments we shall need Lemmas 2 and 3, whose proof is similar to that of Lemma 1.

LEMMA 2. *Suppose that N is a nonnegative real number, a and b are integers, and q is a positive integer. Suppose also that $F(\alpha)$ is a nonnegative real function that can be expressed as a finite Fourier series with nonnegative coefficients $c(\lambda)$. Then the inequalities are valid*:

$$\sum_{|\lambda|\le N} c(a\lambda+b) \le 4q \int_0^1 F(\alpha)\Phi(\alpha)\,d\alpha \le 4q \sum_{|\lambda|\le Nq^{-1}} c(aq\lambda), \tag{4}$$

where

$$\Phi(\alpha) = \sum_{|\lambda|\le Nq^{-1}} \left(1 - \frac{|\lambda|}{[Nq^{-1}]+1}\right) e^{-2\pi i\alpha aq\lambda} \ge 0.$$

PROOF. Let us prove the lemma in the case $a = 1$. For arbitrary a, the proof is the same. Let $N_1 = [Nq^{-1}] + 1$. Then

$$\sum_{|\lambda|\le N} c(\lambda+b) \le \sum_{\mu=0}^{q-1} \sum_{|\lambda|\le N_1} c(q\lambda+\mu+b) = \sum_{\mu=0}^{q-1} \sigma(\mu), \tag{5}$$

where

$$\sigma(\mu) = \sum_{|\lambda|\le N_1} c(q\lambda+\mu+b).$$

Let us estimate $\sigma(\mu)$:

$$\begin{aligned}
\sigma(\mu) &= \frac{1}{N_1^2} \sum_{x,y=1}^{N_1} \sum_{|\lambda+x-y|\le N_1} c(q(\lambda+x-y)+\mu+b) \le \frac{1}{N_1^2} \sum_{x,y=1}^{N_1} \sum_{|\lambda|\le 2N_1-1} c(q(\lambda+x-y)+\mu+b) \\
&= \frac{1}{N_1^2} \sum_{x,y=1}^{N_1} \sum_{|\lambda|\le 2N_1-1} \int_0^1 F(\alpha) e^{-2\pi i\alpha(q(\lambda+x-y)+\mu+b)}\,d\alpha \\
&= \frac{1}{N_1^2} \int_0^1 F(\alpha) \left|\sum_{x=1}^{N_1} e^{-2\pi i\alpha qx}\right|^2 \sum_{|\lambda|\le 2N_1-1} e^{-2\pi i\alpha(q\lambda+\mu+b)}\,d\alpha.
\end{aligned}$$

Estimating the last sum trivially, we obtain

$$\begin{aligned}
\sigma(\mu) &\le \frac{4}{N_1} \int_0^1 F(\alpha) \left|\sum_{x=1}^{N_1} e^{-2\pi i\alpha qx}\right|^2 d\alpha \\
&= \frac{4}{N_1} \int_0^1 F(\alpha) \sum_{|\lambda|\le N_1-1} (N_1 - |\lambda|) e^{-2\pi i\alpha q\lambda}\,d\alpha = 4\int_0^1 F(\alpha)\Phi(\alpha)\,d\alpha.
\end{aligned}$$

Hence we have

$$\sigma(\mu) \le 4\int_0^1 F(\alpha)\Phi(\alpha)\,d\alpha = 4 \sum_{|\lambda|\le N_1-1} \left(1 - \frac{|\lambda|}{N_1}\right) c(q\lambda) \le 4 \sum_{|\lambda|\le Nq^{-1}} c(q\lambda).$$

Substituting the inequalities obtained into (5), we obtain the assertion of the lemma. □

Corollary. *Let the assumptions of Lemma 2 and $Nq^{-1} = O(1)$ be satisfied. Then the following estimate is valid*:

$$\sum_{|\lambda| \le Nq^{-1}} c(aq\lambda) \ll c(0). \tag{6}$$

Proof. Substituting $Nq^{-1}, aq, N_1, 0$ for N, a, q, b, respectively, in the estimate (4), we obtain

$$\sum_{|\lambda| \le Nq^{-1}} c(aq\lambda) \le 4N_1 \sum_{|\lambda| \le N(qN_1)^{-1}} c(aqN_1\lambda) = 4N_1 c(0) \ll c(0). \quad \square$$

The proofs of Lemma 3 and its corollary are similar to the one above.

Lemma 3. *For any integers $a_{1,1}, a_{1,2}, \dots, a_{n,n}, b_1, \dots, b_n$ and positive integers $q_1, \dots, q_n$, let*

$$l_\nu = \sum_{j=1}^{n} a_{\nu,j}\lambda_j, \quad l'_\nu = \sum_{j=1}^{n} a_{\nu,j}q_j\lambda_j, \qquad \nu = 1, \dots, n.$$

Next, suppose that $N_1, \dots, N_n$ are nonnegative real numbers and $F(\alpha_1, \dots, \alpha_n)$ is a nonnegative real function expressed as a finite Fourier series with nonnegative coefficients $c(\lambda_1, \dots, \lambda_n)$. Then the following estimate is valid:

$$\begin{aligned} \sum_{|\lambda_1| \le N_1} \cdots \sum_{|\lambda_n| \le N_n} c(l_1 + b_1, \dots, l_n + b_n) &\le 4^n q_1 \cdots q_n \int_0^1 \cdots \int_0^1 F(\alpha_1, \dots, \alpha_n)\Phi \, d\alpha_1 \cdots d\alpha_n \\ &\le 4^n q_1 \cdots q_n \sum_{|\lambda_1| \le N_1 q_1^{-1}} \cdots \sum_{|\lambda_n| \le N_n q_n^{-1}} c(l'_1, \dots, l'_n), \end{aligned}$$

where

$$\Phi = \sum_{|\lambda_1| \le N_1 q_1^{-1}} \cdots \sum_{|\lambda_n| \le N_n q_n^{-1}} \left(1 - \frac{|\lambda_1|}{[N_1 q_1^{-1}] + 1}\right) \cdots \left(1 - \frac{|\lambda_n|}{[N_n q_n^{-1}] + 1}\right) e^{-2\pi i(\alpha_1 l'_1 + \cdots + \alpha_n l'_n)} \ge 0.$$

Corollary. *Let the assumptions of Lemma 3 be satisfied, and let $N_j q_j^{-1} = O(1)$ for some j, $1 \le j \le n$. In that case, if calculations are carried out up to constants, then it can be assumed that λ_j takes only the zero value.*

§3. Main lemmas

Lemma 4. *Suppose that $1 \le r \le n$, H, Q, $\mu_r, \dots, \mu_{n-1}$, a are integers, and $H' = H - aQ$. Then the system*

$$\left\{\begin{aligned} &x_1 + \cdots - y_k = 0, \\ &\dots\dots\dots\dots\dots\dots \\ &x_1^{r-1} + \cdots - y_k^{r-1} = 0, \\ &C_n^r(x_1^r + \cdots - y_k^r) = \mu_r Q, \\ &C_n^{r+1}(x_1^{r+1} + \cdots - y_k^{r+1}) + \mu_r H = \mu_{r+1} Q, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &C_n^{n-1}(x_1^{n-1} + \cdots - y_k^{n-1}) + \mu_{n-2} H = \mu_{n-1} Q, \\ &C_n^n(x_1^n + \cdots - y_k^n) + \mu_{n-1} H = \mu_n \end{aligned}\right. \tag{7}$$

is equivalent to the system

$$\begin{cases} (x_1+a)+\cdots-(y_k+a)=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ (x_1+a)^{r-1}+\cdots-(y_k+a)^{r-1}=0, \\ C_n^r\big((x_1+a)^r+\cdots-(y_k+a)^r\big)=\mu_r'Q, \\ C_n^{r+1}\big((x_1+a)^{r+1}+\cdots-(y_k+a)^{r+1}\big)+\mu_r'H'=\mu_{r+1}'Q, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_n^{n-1}\big((x_1+a)^{n-1}+\cdots-(y_k+a)^{n-1}\big)+\mu_{n-2}'H'=\mu_{n-1}'Q, \\ C_n^n\big((x_1+a)^n+\cdots-(y_k+a)^n\big)+\mu_{n-1}'H'=\mu_n', \end{cases} \tag{8}$$

where

$$\mu_l'=\sum_{j=r}^{l} C_{n-j-1}^{l-j}a^{l-j}\mu_j, \qquad l=r,\dots,n-1, \quad \mu_n'=\mu_n.$$

Proof. The first $r-1$ equations in system (8) readily follow from system (7). Set $\mu_{r-1}=0$. Let $r\le l\le n-1$. Then

$$\begin{aligned} &C_n^l\big((x_1+a)^l+\cdots-(y_k+a)^l\big)+\mu_{l-1}'H'-\mu_l'Q \\ &\quad = C_n^l\sum_{j=r}^{l} C_l^j a^{l-j}(x_1^j+\cdots-y_k^j)+\mu_{l-1}'H'-\mu_l'Q \\ &\quad = \sum_{j=r}^{l} C_{n-j}^{l-j}a^{l-j}(\mu_jQ-\mu_{j-1}H)+(H-aQ)\sum_{j=r}^{l} C_{n-j-1}^{l-j-1}a^{l-j-1}\mu_j-Q\sum_{j=r}^{l} C_{n-j-1}^{l-j}a^{l-j}\mu_j \\ &\quad = H\sum_{j=r}^{l}(C_{n-j-1}^{l-j-1}a^{l-j-1}\mu_j-C_{n-j}^{l-j}a^{l-j}\mu_{j-1})+Q\sum_{j=r}^{l} a^{l-j}\mu_j(C_{n-j}^{l-j}-C_{n-j-1}^{l-j-1}-C_{n-j-1}^{l-j})=0. \end{aligned}$$

For $l=n$, the proof can be carried out in reverse order in a similar way. $\square$

Remark. If in system (7) $\mu_r,\dots,\mu_{n-1}$ assume integer values independently of one another, then in system (8) the expressions $\mu_r',\dots,\mu_{n-1}'$ also assume integer values independently of one another.

Lemma 5. *Suppose that $1\le r\le n$, H, Q, β are integers, $0\le\beta\le r+1$, p is a prime, $(Q,p)=1$, $H=p^\alpha h$, $(h,p)=1$, $\gamma=\min(1,\alpha)$, and $\alpha\beta=0$. Then the system*

$$\begin{cases} x_1+\cdots-y_k=0, \\ \dots\dots\dots\dots\dots\dots \\ x_1^{r-1}+\cdots-y_k^{r-1}=0, \\ C_n^rp^r(x_1^r+\cdots-y_k^r)=\mu_rQp^\beta, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_n^np^n(x_1^n+\cdots-y_k^n)+\mu_{n-1}Hp^\beta=0 \end{cases} \tag{9}$$

is equivalent to the system

$$\begin{cases} x_1+\cdots-y_k=0, \\ \dots\dots\dots\dots\dots\dots \\ x_1^{r-1}+\cdots-y_k^{r-1}=0, \\ C_n^r(x_1^r+\cdots-y_k^r)=\lambda_rQp^{1-\gamma}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_n^n(x_1^n+\cdots-y_k^n)+\lambda_{n-1}Hp^{-\gamma}=0, \end{cases} \tag{10}$$

where $\mu_r, \dots, \mu_{n-1}$ *and* $\lambda_r, \dots, \lambda_{n-1}$ *are related by*

$$\mu_r = p^{(r+1)-\beta-\gamma}\lambda_r, \quad \dots, \quad \mu_{n-1} = p^{n-\beta-\gamma}\lambda_{n-1}. \tag{11}$$

Proof. relations (9) that $\mu_r, \dots, \mu_{n-1}$ must be of the form (11). After the substitution of these expressions into system (9) and cancellations we obtain system (10). The arguments in reverse order prove that relations (10) yield system (9). $\square$

Lemma 6. *Consider the system of congruences*

$$\begin{cases} x_1 + \dots - y_n \equiv \lambda_1 \pmod{p}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_1^r + \dots - y_n^r \equiv \lambda_r \pmod{p^r} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_1^n + \dots - y_n^n \equiv \lambda_n \pmod{p^r}, \end{cases} \qquad \begin{gathered} 0 \le x_1, \dots, y_n \le Np^r - 1, \\ x_s \not\equiv x_t \pmod{p}, \quad s \ne t, \quad 1 \le s, t \le n, \quad p \ge 4n. \end{gathered}$$

If $T(\lambda_1, \dots, \lambda_n)$ *is the number of solutions of this system, then the inequalities are valid*:

$$\frac{4}{9}N^{2n}p^{rn+r(r-1)/2} \le T = T(0, \dots, 0) \le n!N^{2n}p^{rn+r(r-1)/2}.$$

The proof of these is contained in [3].

§4. The number of solutions to Hardy's equation and Waring's

Consider the positive integers

$$P_0 = P, \quad P_1 = [P_0^{1/2}] + 1, \quad \dots, \quad P_{n-1} = [P_{n-2}^{(n-1)/n}] + 1.$$

Choose primes $p_1, \dots, p_{n-1}$ so that the inequalities are valid:

$$P_0^{1/2} \le p_1 < 2P_0^{1/2}, \quad P_1^{1/3} \le p_2 < 2P_1^{1/3}, \quad \dots, \quad P_{n-2}^{1/n} \le p_{n-1} < 2P_{n-2}^{1/n}.$$

For $\nu = 1, \dots, n-1$, the relations are valid:

$$p_\nu P_\nu \ge P_{\nu-1}, \qquad P_\nu \asymp P^{1/(\nu+1)} \asymp p_\nu^\nu, \qquad p_1 \cdots p_\nu \asymp P^{1-1/(\nu+1)} \asymp P_\nu^\nu.$$

Let $0 \le r \le n-1$. Set $Q_r = p_1 \cdots p_r$ ($Q_0 = 1$). By $T_{k,r}(P')$ denote the number of solutions of the system

$$\begin{cases} x_1 + \dots - y_k = 0, \\ \dots\dots\dots\dots\dots \\ x_1^r + \dots - y_k^r = 0, \\ C_n^{r+1}(x_1^{r+1} + \dots - y_k^{r+1}) = \mu_{r+1}Q_r, \\ C_n^{r+2}(x_1^{r+2} + \dots - y_k^{r+2}) + \mu_{r+1}z_r = \mu_{r+2}Q_r, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_n^{n-1}(x_1^{n-1} + \dots - y_k^{n-1}) + \mu_{n-2}z_r = \mu_{n-1}Q_r, \\ C_n^n(x_1^n + \dots - y_k^n) + \mu_{n-1}z_r = 0, \end{cases}$$
$$0 \le x_1, \dots, y_k < P', \quad 0 \le z_r < Q_r,$$

in which $\mu_{r+1}, \dots, \mu_{n-1}$ may take arbitrary integer values. For $r = 0$, we obtain the relation $T_{k,0}(P_0) = I_k(P)$.

THEOREM 1. *Suppose that* $n \geq 2$, $k \geq n^2/2$, $1 \leq r \leq n-1$, *and* $p_r \geq 4n$. *Then*

$$T_{k,r-1}(P_{r-1}) \ll p_r^{2(k-n)-1-r(r+1)/2} P_{r-1}^{2n} T_{k-n,r}(P_r).$$

PROOF. Let us denote the integral over the unit n-dimensional cube as follows:

$$\int_0^1 \cdots \int_0^1 F(\alpha_1, \dots, \alpha_n)\, d\alpha_1 \cdots d\alpha_n = \int_{E_n} F(\alpha_1, \dots, \alpha_n)\, d\overline{\alpha}.$$

Let

$$f(x) = \alpha_1 x + \cdots + \alpha_{r-1} x^{r-1} + \alpha_r C_n^r x^r + \cdots + \alpha_n C_n^n x^n,$$
$$S(a) = \sum_{x=0}^{P_r - 1} e^{2\pi i f(a + p_r x)}, \qquad S = \sum_{a=0}^{p_r - 1} S(a).$$

According to Lemma 3, we have

$$T_{k,r-1}(P_{r-1}) \leq T_{k,r-1}(p_r P_r) \ll \sum_{z_{r-1}=0}^{Q_{r-1}-1} \int_{E_n} |S|^{2k} \Phi\, d\overline{\alpha}, \tag{12}$$

where

$$\Phi = \sum_{\mu_r, \dots, \mu_{n-1}} \left(1 - \frac{|\mu_r|}{N_r}\right) \cdots \left(1 - \frac{|\mu_{n-1}|}{N_{n-1}}\right) e^{-2\pi i (\alpha_r \mu_r Q_{r-1} + \cdots + \alpha_n(-\mu_{n-1} z_{r-1}))} \geq 0$$

with some nonnegative $N_r, \dots, N_{n-1}$. Set $z_r = z_{r-1} + aQ_{r-1}$. For each z_{r-1}, there exists exactly one value of a for which $(z_r, p_r) = p_r$. Denote this value by a_0. Moreover,

$$|S|^{2k} = \left| S(a_0) + \sum_{a \neq a_0} S(a) \right|^{2k} \ll |S(a_0)|^{2k} + \left| \sum_{a \neq a_0} S(a) \right|^{2k},$$
$$T_{k,r-1}(P_{r-1}) \ll T_1 + T_2, \tag{13}$$

where

$$T_1 = \sum_{z_{r-1}} \int_{E_n} |S(a_0)|^{2k} \Phi\, d\overline{\alpha}, \qquad T_2 = \sum_{z_{r-1}} \int_{E_n} \left| \sum_{a \neq a_0} S(a) \right|^{2k} \Phi\, d\overline{\alpha}.$$

Expressing Φ as a sum, taking the summation sign outside the integral sign, and discarding factors of the form

$$\left(1 - \frac{|\mu_r|}{N_r}\right) \cdots \left(1 - \frac{|\mu_{n-1}|}{N_{n-1}}\right),$$

we see that the value of T_1 does not exceed the number of solutions of the system

$$\begin{cases} (a_0 + p_r x_1) + \cdots - (a_0 + p_r y_k) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (a_0 + p_r x_1)^{r-1} + \cdots - (a_0 + p_r y_k)^{r-1} = 0, \\ C_n^r \big((a_0 + p_r x_1)^r + \cdots - (a_0 + p_r y_k)^r \big) = \mu_r Q_{r-1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_n^n \big((a_0 + p_r x_1)^n + \cdots - (a_0 + p_r y_k)^n \big) + \mu_{n-1} z_{r-1} = 0, \end{cases}$$
$$0 \leq x_1, \dots, y_k < P_r, \quad 0 \leq z_{r-1} < Q_{r-1}.$$

Using successively Lemmas 4 and 5, we see that T_1 does not exceed the number of solutions of the system

$$\begin{cases} x_1 + \cdots - y_k = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1^{r-1} + \cdots - y_k^{r-1} = 0, \\ C_n^r(x_1^r + \cdots - y_k^r) = \mu_r Q_{r-1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_n^n(x_1^n + \cdots - y_k^n) + \mu_{n-1} z_r p_r^{-1} = 0, \end{cases}$$
$$0 \le x_1, \dots, y_k < P_r, \quad z_r = z_{r-1} + a_0 Q_{r-1}, \quad 0 \le z_{r-1} < Q_{r-1}.$$

According to Lemma 3, we have $T_1 \ll p_r^{n-r} T_3$, where T_3 is the number of solutions of the system

$$\begin{cases} x_1 + \cdots - y_k = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1^{r-1} + \cdots - y_k^{r-1} = 0, \\ C_n^r(x_1^r + \cdots - y_k^r) = \mu_r Q_r, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_n^n(x_1^n + \cdots - y_k^n) + \mu_{n-1} z_r = 0 \end{cases} \tag{14}$$

under the same constraints on the s. Since $p_1 \cdots p_r \asymp P_r^r$, $\mu_r = O(1)$ and, by the corollary of Lemma 3, we can assume that $\mu_r = 0$. Discarding the condition $(z_r, p_r) = p_r$, we obtain the relation

$$T_1 \ll p_r^{n-r} T_{k,r}(P_r) \ll p_r^{n-r} P_r^{2n} T_{k-n,r}(P_r). \tag{15}$$

Let us now estimate the value T_2. Let us transform the integrand appearing in the definition of T_2:

$$\left|\sum_{a \ne a_0} S(a)\right|^{2k} = \left|\sum_{a_1 \ne a_0} \cdots \sum_{a_k \ne a_0} S(a_1) \cdots S(a_k)\right|^2. \tag{16}$$

We shall assign the collection $a_1, \dots, a_k$ to the first class if among the numbers $a_1, \dots, a_k$ we can find n distinct ones. All the other collections will be assigned to the second class. Let us divide the multiple sum on the right-hand side of the relation (16) into two sums σ_1 and σ_2 over all collections belonging to the first and second classes, respectively. Moreover,

$$\left|\sum_{a \ne a_0} S(a)\right|^{2k} = |\sigma_1 + \sigma_2|^2 \ll |\sigma_1|^2 + |\sigma_2|^2, \qquad T_2 \ll T_4 + T_5, \tag{17}$$

where

$$T_4 = \sum_{z_{r-1}} \int_{E_n} |\sigma_1|^2 \Phi \, d\overline{\alpha}, \qquad T_5 = \sum_{z_{r-1}} \int_{E_n} |\sigma_2|^2 \Phi \, d\overline{\alpha}.$$

Each summand in the sum over all collections of the first class can be rearranged so that n distinct numbers from the collection $a_1, \dots, a_k$ occupy the first n places. Therefore,

$$\begin{aligned} T_4 &\ll \sum_{z_{r-1}} \int_{E_n} \left| \sum_{a_1, \dots, a_n}^{*} S(a_1) \cdots S(a_n) \right|^2 \left| \sum_{a \ne a_0} S(a) \right|^{2(k-n)} \Phi \, d\overline{\alpha} \\ &\ll p_r^{2(k-n)-1} \sum_{z_{r-1}} \int_{E_n} \left| \sum_{a_1, \dots, a_n}^{*} S(a_1) \cdots S(a_n) \right|^2 \sum_{a \ne a_0} |S(a)|^{2(k-n)} \Phi \, d\overline{\alpha}, \end{aligned}$$

where summation in $\sum^*$ is taken over all collections $a_1, \ldots, a_n$ in which $a_s \neq a_t$ for $s \neq t$, $1 \leq s, t \leq n$. Treating the function Φ, as above, we see that

$$T_4 \ll p_r^{2(k-n)-1} T_6, \tag{18}$$

where T_6 is the number of solutions of the system

$$\begin{cases} x_1 + \cdots - y_n + (a + p_r x_{n+1}) + \cdots - (a + p_r y_k) = 0, \\ \dotfill \\ x_1^{r-1} + \cdots - y_n^{r-1} + (a + p_r x_{n+1})^{r-1} + \cdots - (a + p_r y_k)^{r-1} = 0, \\ C_n^r \big(x_1^r + \cdots - y_n^r + (a + p_r x_{n+1})^r + \cdots - (a + p_r y_k)^r \big) = \mu_r Q_{r-1}, \\ \dotfill \\ C_n^n \big(x_1^n + \cdots - y_n^n + (a + p_r x_{n+1})^n + \cdots - (a + p_r y_k)^n \big) + \mu_{n-1} z_{r-1} = 0, \end{cases}$$
$$0 \leq x_1, \ldots, y_n < p_r P_r, \qquad x_s \not\equiv x_t \ (\operatorname{mod} p_r), \quad y_s \not\equiv y_t \ (\operatorname{mod} p_r) \quad \text{for } s \neq t,$$
$$0 \leq x_{n+1}, \ldots, y_k < P_r, \quad 0 \leq a < p_r, \quad a \neq a_0, \quad 0 \leq z_{r-1} < Q_{r-1}.$$

According to Lemma 3, we have

$$T_6 \ll p_r^{(r+1)(n-r)} T_7, \tag{19}$$

where T_7 is the number of solutions of the system

$$\begin{cases} x_1 + \cdots - y_n + (a + p_r x_{n+1}) + \cdots - (a + p_r y_k) = 0, \\ \dotfill \\ x_1^{r-1} + \cdots - y_n^{r-1} + (a + p_r x_{n+1})^{r-1} + \cdots - (a + p_r y_k)^{r-1} = 0, \\ C_n^r \big(x_1^r + \cdots - y_n^r + (a + p_r x_{n+1})^r + \cdots - (a + p_r y_k)^r \big) = \mu_{r+1} Q_{r-1} p_r^{r+1}, \\ \dotfill \\ C_n^n \big(x_1^n + \cdots - y_n^n + (a + p_r x_{n+1})^n + \cdots - (a + p_r y_k)^n \big) + \mu_{n-1} z_{r-1} p_r^{r+1} = 0 \end{cases}$$

under the same constraints on the variables. According to Lemma 4, the last system can be rewritten in the form

$$\begin{cases} (x_1 - a) + \cdots - (y_n - a) + p_r (x_{n+1} + \cdots - y_k) = 0, \\ \dotfill \\ (x_1 - a)^{r-1} + \cdots - (y_n - a)^{r-1} + p_r^{r-1} (x_{n+1}^{r-1} + \cdots - y_k^{r-1}) = 0, \\ C_n^r \big((x_1 - a)^r + \cdots - (y_n - a)^r + p_r^r (x_{n+1}^r + \cdots - y_k^r) \big) = \mu_r Q_{r-1} p_r^{r+1}, \\ \dotfill \\ C_n^n \big((x_1 - a)^n + \cdots - (y_n - a)^n + p_r^n (x_{n+1}^n + \cdots - y_k^n) \big) + \mu_{n-1} z_r p_r^{r+1} = 0, \end{cases}$$

where $z_r = z_{r-1} + a Q_{r-1}$ and the variables vary within the same limits. Denote by $N_n^*(\lambda_1, \ldots, \lambda_n)$ the number of solutions of the system

$$\begin{cases} x_1 + \cdots - y_n = \lambda_1, \\ \dotfill \\ x_1^n + \cdots - y_n^n = \lambda_n, \end{cases} \qquad \begin{gathered} -a \leq x_1, \ldots, y_n < p_r P_r - a, \\ x_s \not\equiv x_t \ (\operatorname{mod} p_r), \quad y_s \not\equiv y_t \ (\operatorname{mod} p_r) \quad \text{for } s \neq t. \end{gathered}$$

Then we can write

$$T_7 = \sum_{\lambda_1, \ldots, \lambda_n} N_n^*(\lambda_1 p_r, \ldots, \lambda_{r+1} p_r^{r+1}, \ldots, \lambda_n p_r^{r+1}) T_8(\lambda_1 p_r, \ldots, \lambda_{r+1} p_r^{r+1}, \ldots, \lambda_n p_r^{r+1}),$$

where $T_8(\lambda_1,\ldots,\lambda_n)$ is the number of solutions of the system

$$\begin{cases} p_r(x_1+\cdots-y_{k-n})+\lambda_1=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_r^{r-1}(x_1^{r-1}+\cdots-y_{k-n}^{r-1})+\lambda_{r-1}=0, \\ C_n^r\big(p_r^r(x_1^r+\cdots-y_{k-n}^r)+\lambda_r\big)=\mu_rQ_{r-1}p_r^{r+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_n^n\big(p_r^n(x_1^n+\cdots-y_{k-n}^n)+\lambda_n\big)+\mu_{n-1}z_rp_r^{r+1}=0, \end{cases}$$
$$0\le x_1,\ldots,y_{k-n}<P_r, \qquad 0\le z_r<Q_r, \quad (z_r,p_r)=1.$$

Let $T_8=T_8(0,\ldots,0)$. In view of Lemma 6, we can write

$$T_7\le T_8\sum_{\lambda_1,\ldots,\lambda_n}N_n^*(\lambda_1p_r,\ldots,\lambda_{r+1}p_r^{r+1},\ldots,\lambda_np_r^{r+1})\ll P_{r-1}^{2n}p_r^{-r(r+1)/2-(r+1)(n-r)}T_8. \tag{20}$$

According to Lemma 5, T_8 does not exceed the number of solutions of the system

$$\begin{cases} x_1+\cdots-y_{k-n}=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{r-1}+\cdots-y_{k-n}^{r-1}=0, \\ C_n^r(x_1^r+\cdots-y_{k-n}^r)=\mu_rQ_r, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_n^n(x_1^n+\cdots-y_{k-n}^n)+\mu_{n-1}z_r=0 \end{cases}$$

under the same constraints on the variables. Since $p_1\cdots p_r\asymp P_r^r$, we have $\mu_r=O(1)$. By the corollary of Lemma 3 we can take $\mu_r=0$. Discarding the condition $(z_r,p_r)=1$, we obtain the inequality

$$T_8\ll T_{k-n,r}(P_r). \tag{21}$$

Combining the estimates (18)–(21), we obtain

$$T_4\ll p_r^{2(k-n)-1-r(r+1)/2}P_{r-1}^{2n}T_{k-n,r}(P_r). \tag{22}$$

Let us estimate the number T_5. The number of collections of the second class does not exceed $n^kp_r^{n-1}$. Treating the function Φ as above, we see that

$$T_5\ll p_r^{2n-2}\sum_{z_{r-1}}\sum_{a\ne a_0}\int_{E_n}|S(a)|^{2k}\Phi\,d\overline{\alpha}\ll p_r^{2n-2}P_r^{2n}\sum_{z_{r-1}}\sum_{a\ne a_0}\int_{E_n}|S(a)|^{2(k-n)}\Phi\,d\overline{\alpha}\ll p_r^{2n-2}P_r^{2n}T_9,$$

where T_9 is the number of solutions of the system

$$\begin{cases} (a+p_rx_1)+\cdots-(a+p_ry_{k-n})=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (a+p_rx_1)^{r-1}+\cdots-(a+p_ry_{k-n})^{r-1}=0, \\ C_n^r\big((a+p_rx_1)^r+\cdots-(a+p_ry_{k-n})^r\big)=\mu_rQ_r, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_n^n\big((a+p_rx_1)^n+\cdots-(a+p_ry_{k-n})^n\big)+\mu_{n-1}z_{r-1}=0, \end{cases}$$
$$0\le x_1,\ldots,y_{k-n}<P_r, \quad 0\le z_{r-1}<Q_{r-1}, \quad 0\le a<p_r, \quad a\ne a_0.$$

Using Lemmas 4 and 5 in succession, we see that T_9 does not exceed the number of solutions of the system

$$\begin{cases} x_1 + \cdots - y_{k-n} = 0, \\ \dots\dots\dots\dots\dots\dots \\ x_1^{r-1} + \cdots - y_{k-n}^{r-1} = 0, \\ C_n^r(x_1^r + \cdots - y_{k-n}^r) = \mu_r Q_r, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_n^n(x_1^n + \cdots - y_{k-n}^n) + \mu_{n-1} z_r = 0 \end{cases}$$

under the same constraints on the variables. Again replacing μ_r by 0, we obtain the estimates

$$T_9 \ll T_{k-n,r}(P_r), \qquad T_5 \ll p_r^{2n-2} P_r^{2n} T_{k-n,r}(P_r). \tag{23}$$

the estimates (15), (22), and (23) that

$$T_{k,r-1}(P_{r-1}) \ll \big(p_r^{n-r} P_r^{2n} + p_r^{2(k-n)-1-r(r+1)/2} P_{r-1}^{2n} + p_r^{2n-2} P_r^{2n}\big) T_{k-n,r}(P_r). \tag{24}$$

Since $k \ge n^2/2$, in (24) the second summand in parentheses is the leading term. Therefore,

$$T_{k,r-1}(P_{r-1}) \ll p_r^{2(k-n)-1-r(r+1)/2} P_{r-1}^{2n} T_{k-n,r}(P_r). \quad \square$$

THEOREM 2. *Suppose that $n \ge 2$, $k \ge n^2/2$ and the estimate*

$$N_k(P) \ll P^{2k-n(n+1)/2+\varepsilon_0}.$$

holds. Then, for $k_1 \ge n(n-1)+k$, the following estimate is valid:

$$I_{k_1}(P) \ll P^{2k_1-n+\varepsilon_0/n}.$$

PROOF. It suffices to prove the theorem for $k_1 = n(n-1)+k$. Let us assume that $P \ge (4n)^{n^2}$. This ensures the validity of the inequality $p_{n-1} \ge 4n$. Let us show that for $\tau = 0, \dots, n-1$ the following estimate is valid:

$$T_{k+n\tau,n-\tau-1}(P_{n-\tau-1}) \ll P_{n-\tau-1}^{2(k+n\tau)-n(n+1)/2+\tau(\tau+1)/2} p_1 \cdots p_{n-\tau-1} P^{\varepsilon_0/n}. \tag{25}$$

For $\tau = 0$, we must estimate the number $T_{k,n-1}(P_{n-1})$ that is equal to the number of solutions of the system

$$\begin{cases} x_1 + \cdots - y_k = 0, \\ \dots\dots\dots\dots\dots\dots \\ x_1^{n-1} + \cdots - y_k^{n-1} = 0, \\ C_n^n(x_1^n + \cdots - y_k^n) = 0, \end{cases} \qquad 0 \le x_1, \dots, y_k < P_{n-1}, \quad 0 \le z_{n-1} < p_1 \cdots p_{n-1}.$$

The variable z_{n-1} does not appear in the system; therefore,

$$T_{k,n-1}(P_{n-1}) \ll N_k(P_{n-1}) p_1 \cdots p_{n-1} \ll P_{n-1}^{2k-n(n+1)/2} p_1 \cdots p_{n-1} P^{\varepsilon_0/n}.$$

Let the estimate (25) be valid for some τ, $0 \le \tau \le n-2$. Let us prove its validity for $\tau+1$. By Theorem 1, we have

$$T_{k+n(\tau+1),n-\tau-2}(P_{n-\tau-2}) \ll p_{n-\tau-1}^{2(k+n\tau)-1-\frac{(n-\tau-1)(n-\tau)}{2}} P_{n-\tau-2}^{2n} T_{k+n\tau,n-\tau-1}(P_{n-\tau-1}).$$

Using inequality (25), we obtain

$$T_{k+n(\tau+1),n-\tau-2}(P_{n-\tau-2}) \ll P_{n-\tau-2}^{2(k+n(\tau+1))-\frac{n(n+1)}{2}+\frac{(\tau+1)(\tau+2)}{2}} p_1 \cdots p_{n-\tau-2} P^{\varepsilon_0/n}.$$

For $\tau = n-1$, formula (25) yields the estimate

$$I_{k_1}(P) = T_{k_1,0}(P_0) \ll P_0^{2k_1-n+\varepsilon_0/n}. \quad \square \tag{26}$$

Corollary. *Suppose that* $n \geq 2$, $\tau \geq n/2$, *and* $k \geq n(n-1) + n\tau$. *Then*

$$I_k(P) \ll P^{2k-n+\frac{n}{2}\left(1-\frac{1}{n}\right)^{\tau}}. \tag{27}$$

The proof follows directly from Theorem 2 and the estimate (1).

Remark. The same method can be used to obtain estimates of the number of solutions and asymptotic formulas for systems of the form

$$x_1^{\nu} + \cdots - y_k^{\nu} = 0, \qquad 0 \leq x_1, \ldots, y_k < P, \quad \nu = r_1, r_2, \ldots, r_s, \quad 1 \leq r_1 < r_2 < \cdots < r_s = n.$$

Thus, for example, for the number $I_{k,r,n}(P)$ equal to the number of solutions of the system

$$\begin{cases} x_1^r + \cdots - y_k^r = 0, \\ x_1^n + \cdots - y_k^n = 0, \end{cases} \qquad 0 \leq x_1, \ldots, y_k < P,$$

where $n \geq 2$, $\tau \geq n/2$, $k \geq n(n-1) + n\tau$, the following estimate is valid:

$$I_{k,n,r}(P) \ll P^{2k-r-n+\frac{rn}{2}\left(1-\frac{1}{n}\right)^{\tau}}.$$

Theorem 3. *Let* $n \geq 4$. *Then the asymptotic formula* (3) *is valid for* $k \geq 2[n^2(\ln n + \ln\ln n + 6)]$.

The proof of the theorem is similar to that of Theorem 1 from [1, Chap. 6] with the substitution of (27) for (1) in estimating the integral over the region of the second class.

The author wishes to thank N. M. Korobov for setting the problem and supervising his work.

References

1. I. M. Vinogradov, *The Method of Trigonometric Sums in Number Theory* [in Russian], Nauka, Moscow (1980).
2. V. A. Bykovskii, "On systems of inequalities," *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)*, **129**, 3–33 (1981).
3. A. A. Karatsuba, "On systems of congruences," *Izv. Akad. Nauk SSSR Ser. Mat.* [*Math. USSR-Izv.*], **29**, 935–944 (1965).

M. V. Lomonosov Moscow State University

Translated by N. K. Kulman