Far Eastern Mathematical Journal. 2024. V. 24. No 2. P. 193–199

UDC 517.958
MSC2020 35Q20 +35Q60

© E. O. Kovalenko[1], A. A. Sushchenko[1,2]

Focusing of hydroacoustic images based on multiangle sounding data

In this paper we prove a convergent part of inhomogeneous Groshev type theorem for non–degenerate curves in Euclidean space where an error function is not necessarily monotonic. Our result naturally incorporates and generalizes the homogeneous measure theorem for non-degenerate curves. In particular, the method of Inhomogeneous Transference Principle and Sprindzuk's method of essential and inessential domains are used in the proof.

Key words: *Inhomogeneous Diophantine approximation, Khintchine theorem, non-degenerate curve.*

DOI: https://doi.org/10.47910/FEMJ202418

Introduction

In this paper, the problem of improving the quality and constructing of sonar images of the seabed based on measurements of a side-scan sonar (SSS) is considered. It is assumed that the carrier of the receiving-transmitting antenna, emitting a pulsed signal, moves at a constant speed along a straight line. As a mathematical model, the equation for the transfer of high-frequency acoustic radiation [1–4] with a boundary condition describing diffuse reflection on the bottom surface [5–9] is used. Reconstruction of a bottom scattering coefficient posed as an inverse problem for this model. In the framework of the single-scattering approximation an integral equation was obtained. The equation has an explicit solution only for a narrow receiving antenna radiation pattern [5,6]. With an increase in the width of the radiation pattern, the use of an explicit formula for inverse problem solution leads to bluring of bottom objects on the sonar images.

To overcome this defect, one can solve the integral equation, for example, by discretizing the continuous problem and reducing it to solving a system of linear algebraic equations. However, the conventional sounding method using two single-beam SSS located on different sides of the carrier [3–7] leads to an ill-posed system of linear algebraic

[1] Institute for Applied Mathematics, Far Eastern Branch of the Russian Academy of Sciences, Russia, 690041, Vladivostok, Radio st., 7.
[2] Far Eastern Federal University, Russia, 690922, Vladivostok, Russky Island, Ajax Bay 10.
E-mail: kovalenko.eo@dvfu.ru (E. O. Kovalenko), sushchenko.aa@dvfu.ru (A. A. Sushchenko).

equations. The solution of the problem becomes sensitive to errors in the initial data, and, as a result, it is impossible to obtain an image of acceptable quality of the seabed even for slightly noisy data [7].

In this paper, we consider the case of multipath scanning when a multi-beam antenna receives an echolocation signal from various angular directions. Note that a similar measurement scheme can be implemented with a single-beam detector by increasing the number of traverses [10].

1 Direct and inverse problems for the nonstationary radiative transfer equation

The nonstationary radiative transfer equation is considered [1–9, 11–15]

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{k}\cdot\nabla_r + \mu\right) I(\mathbf{r},\mathbf{k},t) = \frac{\sigma}{4\pi}\int\limits_{\Omega} I(\mathbf{r},\mathbf{k}',t)d\mathbf{k}' + J(\mathbf{r},\mathbf{k},t), \tag{1}$$

where $\mathbf{r}\in G\subset\mathbb{R}^3$, $t\in[0,T]$ and wave vector $\mathbf{k}$ belongs to the unit sphere $\Omega=\{\mathbf{k}\in\mathbb{R}^3 : |\mathbf{k}|=1\}$. The function $I(\mathbf{r},\mathbf{k},t)$ is the energy flux density of a wave propagating in direction $\mathbf{k}$ with the sound speed c at the time t at the point $\mathbf{r}$. The functions μ and σ are attenuation and the scattering coefficients, and the function J describes the sources of the sound field.

The area G is the upper half-space bounded by the horizontal plane $\gamma=\{\mathbf{r}=(r_1,r_2,r_3)\in\mathbb{R}^3 : r_3=-l\}$, $l>0$. We add to equation (1) the initial and boundary conditions [8, 9]

$$I^-(\mathbf{r},\mathbf{k},t)|_{t<0}=0, \qquad (\mathbf{r},\mathbf{k})\in G\times\Omega, \tag{2}$$

$$I^-(\mathbf{y},\mathbf{k},t)=\frac{\sigma_d(\mathbf{y})}{\pi}\int\limits_{\Omega_+}|\mathbf{n}\cdot\mathbf{k}'|I^+(\mathbf{y},\mathbf{k}',t)d\mathbf{k}', \qquad (\mathbf{y},\mathbf{k},t)\in\Gamma^-. \tag{3}$$

In relations (2), (3) we use the notation $I^{\pm}(\mathbf{y},\mathbf{k},t)=\lim\limits_{\varepsilon\to-0} I(\mathbf{y}\pm\varepsilon\mathbf{k},\mathbf{k},t\pm\varepsilon/c)$,

$$\Gamma^{\pm}=\{(\mathbf{y},\mathbf{k},t)\in\gamma\times\Omega_{\pm}\times(0,T)\}, \quad \Omega_{\pm}=\{\mathbf{k}\in\Omega : \operatorname{sgn}(\mathbf{n}\cdot\mathbf{k})=\pm1\},$$

where $\mathbf{n}=(0,0,-1)$ — is the unit vector of the external normal to the boundary of the domain G. Condition (2) means that there is no radiation in the medium at the initial time, and the boundary condition (3) describes the diffuse reflection on the seabed according to Lambert's cosine law. $\sigma_d(\mathbf{y})$ denotes the bottom scattering coefficient.

Let us pose a problem. **Problem 1.** The equation (1) with conditions (2), (3) for given μ,σ, σ_d, J, c poses an initial-boundary value problem for finding an unknown function I on the set $G\times\Omega\times(0,T)$.

The well-posedness of the direct problem was considered in [9].

We will assume that the function J describes a point impulse sound source moving at the constant speed V in the direction of the axis r_2 and emitting a pulse parcels in time moments $t_0, t_1, \ldots, t_m$

$$J(\mathbf{r},\mathbf{k},t)=\delta(\mathbf{r}-\mathbf{V}t)\sum_{i=1}^{m}\delta(t-t_i), \qquad \mathbf{V}=(0,V,0),\ t_i>0,$$

where δ — is the Dirac delta function. Let complete the system of relation (1)–(3)

$$\int\limits_{\Omega} S_j(\mathbf{k}) I^+(\mathbf{V}t, \mathbf{k}, t) d\mathbf{k} = P_j(t), \quad j = 1, \dots, q, \tag{4}$$

where $S_j(\mathbf{k})$ is nonzero function in subdomain $\Omega_j \subset \Omega$ and denote the directivity pattern of the receiving antenna, and q is the number of sounding tracks of the SSS.

Problem 2. Find the function $\sigma_d(\mathbf{y})$ from relations (1)–(4) for given $\mu, \sigma, J(see(4)), c, P_j, S_j$.

The inverse problem 2 has various physical applications. For example, one arises during the acoustic sounding of the seabed by a SSS moving in a straight direction at a constant speed V, sounding the surrounding space with pulsed signals. The carrier has antennas that measure the total intensity $P_j(t)$ in the sector Ω_j at time moments t. If $q=2$ and the sets $\Omega_1 = \{\mathbf{k} \in \Omega : k_1 < 0\}$, $\Omega_2 = \{\mathbf{k} \in \Omega : k_1 > 0\}$, then we are dealing with the simplest case of the SSS containing one receiving antenna per board [4].

2 Single scattering approximation

The signal intensity P_j at the receiver point at time t in the framework of single scattering approximation in the j-th direction can be represented as the sum of two terms. The first corresponds to the signal reflected from the seabed ($P_{j,\gamma}$), the second one is the signal scattered on the inhomogeneities of the medium ($P_{j,G}$) [3–7]:

$$P_{j,\gamma}(t) = \frac{l^2}{\pi} \frac{\exp(-\mu c(t-t_i))}{t-t_i} \times$$
$$\times \int\limits_0^{2\pi} \frac{\left(S_j\left(\frac{\mathbf{V}t-\mathbf{y}_-}{|\mathbf{V}t-\mathbf{y}|}\right)\sigma_d(\mathbf{y}_-) + S_j\left(\frac{\mathbf{V}t-\mathbf{y}_+}{|\mathbf{V}t-\mathbf{y}|}\right)\sigma_d(\mathbf{y}_+)\right)|\sin\varphi|\cos\theta_i d\varphi}{|y_1||\mathbf{y}-\mathbf{V}t_i|^2|\mathbf{V}t-\mathbf{y}|^2} =$$
$$= \frac{l^2}{\pi} \frac{\exp(-\mu c(t-t_i))}{t-t_i} \int\limits_0^{2\pi} \frac{S_j(\mathbf{k}(\varphi,\theta_i))\sigma_d(\mathbf{y}(\varphi,\theta_i))d\varphi}{|\mathbf{y}-\mathbf{V}t_i|^2|\mathbf{V}t-\mathbf{y}|^2} =$$
$$= \frac{cl^2}{2\pi} \frac{\exp(-\mu c(t-t_i))}{(c(t-t_i)/2)^5} \int\limits_0^{2\pi} S_j(\mathbf{k}(\varphi,\theta_i))\sigma_d(\mathbf{y}(\varphi,\theta_i)d\varphi. \tag{5}$$

$$P_{j,G}(t) = \frac{\sigma c \exp(-\mu c(t-t_i))}{8\pi(c(t-t_i)/2)^2} \int\limits_0^{2\pi} \int\limits_{\theta_i}^{\pi} S_j\left(\mathbf{k}(\theta,\varphi)\right) \sin\theta d\theta d\varphi. \tag{6}$$

Here, θ_i and ϕ denote the zenith direction and the azimuth angle, respectively. And i depends on a number of sounding track of the SSS. The point $\mathbf{y}_\pm$ is defined by $\mathbf{y}_\pm = (\pm|y_1|, y_2, -l)$.

If the radiation pattern of the receiving antenna S_j is narrowly directed in planes perpendicular to the bottom surface $r_3 = -l$: $S_j\left(\mathbf{k}(\theta,\varphi)\right) = \delta(\varphi - \varphi_j)$, where $\delta(\varphi - \varphi_j)$ —

is the Dirac delta function, from formulas (5), (6) the solution to the inverse problem can be obtained as

$$\sigma_{d,j}(\mathbf{y}) = \left(P_j\,(t) - \frac{\sigma c \exp(-2\mu|\mathbf{y}-\mathbf{V}t|)}{8\pi|\mathbf{y}-\mathbf{V}t|^2}\left(1+\frac{l}{|\mathbf{y}-\mathbf{V}t|}\right)\right)\left(\frac{cl^2}{2\pi}\frac{\exp(-2\mu|\mathbf{y}-\mathbf{V}t|)}{|\mathbf{y}-\mathbf{V}t|^5}\right)^{-1},$$

where $t=(y_2+y_1\mathrm{ctg}\varphi_j)/V$.

Obviously, with a narrowly collimated (in the angle φ) radiation pattern, to find the bottom scattering coefficient σ_d it is sufficient to carry out measurements using two receiving antennas, located on different sides of the carrier. For example, when $S_1 = \delta(\varphi-\pi/2)$ and $S_2=\delta(\varphi-3\pi/2)$. This case corresponds to the widely used method of constructing sonar images — successively strip by strip, perpendicular to the movement of the antenna carrier.

3 Numerical algorithm for solving the inverse problem

Solving the inverse problem forms a data set $\sigma_{d,j}(\mathbf{y})$. The following algorithm is applicable to construct a solution that gives the best quality of reconstruction

1. For each number $j=1,\dots,q$ the functions $\sigma_{d,j}(\mathbf{y})$ are calculated using the formulas

$$\sigma_{d,j}(\mathbf{y}) = \left(\widehat{P}_j\,(\mathbf{y}) - \widehat{P}_{j,G}(\mathbf{y})\right)\left(\frac{cl^2c}{2\pi}\frac{\exp(-2\mu\sqrt{y_1^2/\sin^2\varphi_j+l^2})}{(y_1^2/\sin^2\varphi_j+l^2)^{5/2}}\right)^{-1}. \quad (7)$$

Here,

$$\widehat{P}_j\,(\mathbf{y}) = P_j\,(t)\,, \widehat{P}_{j,G}\,(\mathbf{y}) = \frac{\sigma c \exp(-2\mu|\mathbf{y}-\mathbf{V}t|)}{8\pi|\mathbf{y}-\mathbf{V}t|^2}\left(1+\frac{l}{|\mathbf{y}-\mathbf{V}t|}\right).$$

2. The function $\widehat{\sigma}_d(\mathbf{y})$ is constructed by choosing the minimum value from the set $\sigma_{d,1}(\mathbf{y}),\dots,\sigma_{d,q}(\mathbf{y})$, i.e.

$$\widehat{\sigma}_d(\mathbf{y}) = \min_{j=1,\dots,q}\sigma_{d,j}(\mathbf{y}).$$

With a narrow radiation pattern of receiving antennas, all functions $\sigma_{d,j}(\mathbf{y})$ coincide with $\sigma_d(\mathbf{y})$, therefore, the function $\widehat{\sigma}_d(\mathbf{y})$ in this case also coincides with $\sigma_d(\mathbf{y})$. As we will show below in a series of computational experiments, as the width of the radiation pattern increases, the graphic representation of the function $\widehat{\sigma}_d(\mathbf{y})$ fairly well reproduces the structure of the areas γ_i.

4 Numerical experiments

To demonstrate the efficiency of the algorithm for solving the inverse problem, several experiments are carried out. To test the algorithm we use the real data obtained from the SSS during scanning of Zolotoi Rog bay (Vladivostok). In this experiment we use parameters of the environment: $l=12$m, $\mu=0.018\text{m}^{-1}$, $\sigma=0.1\mu$, $t_{i+1}-t_i=0.4$s for any i, $V=1$m/s, $c=1500$m/s. Figure 1 shows data received from a side-scan sonar, here 0 corresponds to the absorbed signal, and 1 corresponds to the reflected signal relative to the received data.

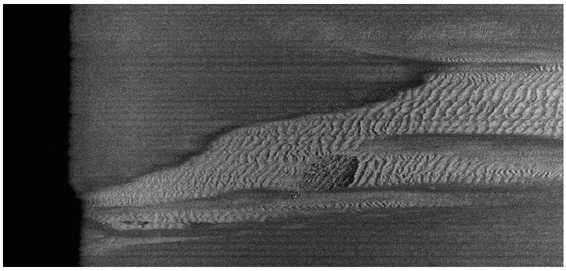
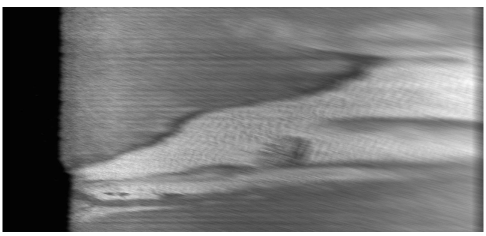


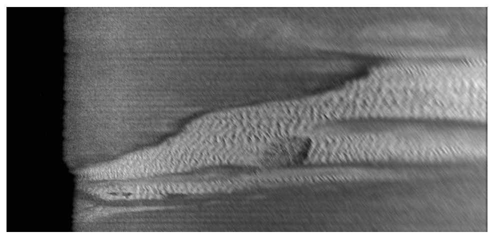
Figure 1: Sea bottom coefficient received from the starboard SSS (reconstructed)

This image was used as a reference value for the sigma coefficient. After that, the signal recorded by the device at different angles was simulated, which was processed using our algorithm

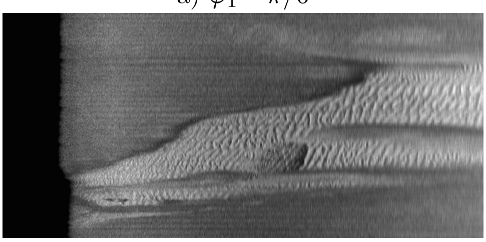
Figures 2a - 2d represent functions $\sigma_{d,j}$ calculated by formula (7) for $\varphi_j = \pi/6, 4\pi/3, \pi/2, 8\pi/3$. The width of the radiation pattern S_j is 0.4 degrees.



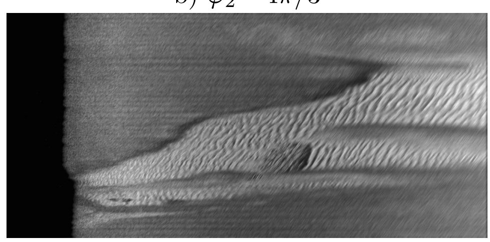
a) $\varphi_1 = \pi/6$



b) $\varphi_2 = 4\pi/3$



c) $\varphi_3 = \pi/2$



d) $\varphi_4 = 8\pi/3$

Figure 2: Graphical representation of the bottom scattering coefficient for different angular directions

To formally assess the quality of reconstruction, we calculated the following metrics: δ_2 — root mean square error, δ_∞ — maximum error, MSE measures the average of the squares of the errors, SSIM is index of structural similarity, PSNR is peak signal-to-noise ratio.

Figure 3 shows the function $\widehat{\sigma}_d(\mathbf{y})$ reconstructed by using the focusing algorithm for $\sigma_{d,j}$ with the radiation pattern width 0.8 degrees. As can be seen in figure 3, the reconstructed image is better than with single-beam reconstruction and the approximation error $\delta_2 = 0.217432, \delta_\infty = 0.49988$.

Table 1: Comparison of relative errors of the data shown in Fig. 2

φ	δ_2	δ_∞	MSE	SSIM	PSNR
$\pi/6$	0.243301	0.588963	0.040541	0.311683	13.921056
$4\pi/3$	0.213612	0.530502	0.016385	0.426709	17.787119
$\pi/2$	0.224288	0.487163	0.014538	0.34398	18.375036
$8\pi/3$	0.218547	0.54073	0.016168	0.390694	17.879247
our method	0.217432	0.49988	0.013667	0.570349	18.609112

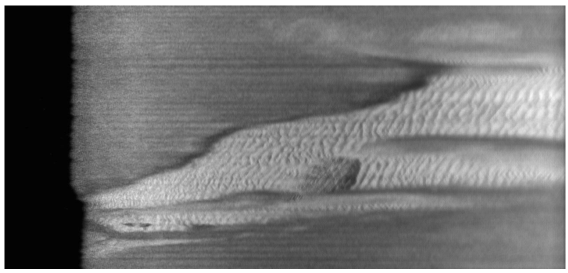


Figure 3: Graphical representation of the function$\widehat{\sigma}_d(\mathbf{y})$ (processed image of the seabed)

Conclusion

In the framework of single-scattering approximation, a new method for solving the inverse problem is proposed.

The reconstruction algorithm which includes different directions for restoration shows acceptable result causes decreasing in the mean square error from 1 to 5 %.

References

[1] Turner J. A., Weaver R. L., “Radiative transfer of ultrasound”, *The Journal of the Acoustical Society of America*, **96**:6, (1994), 3654–3674.

[2] Quijano J. E., Zurk L. M., “Radiative transfer theory applied to ocean bottom modeling”, *The Journal of the Acoustical Society of America*, **126**:4, (2009), 1711–1723.

[3] Prokhorov I. V., Zolotarev V. V., Agafonov I. B., “The problem of acoustic sounding in a fluctuating ocean”, *Dal'nevost. Mat. Zh.*, **11**:1, (2011), 76–87.

[4] Prokhorov I. V., Sushchenko A. A., “Studying the problem of acoustic sounding of the seabed using methods of radiative transfer theory”, *Acoustic journal*, **61**:3, (2015), 400–408.

[5] Prokhorov I. V., Sushchenko A. A., KanV. A., “On the problem of reconstructing the floor topography of a fluctuating ocean”, *Journal of Applied and Industrial Mathematics*, **9**:3, (2015), 412–422.

[6] Kovalenko E. O., Sushchenko A. A., Prokhorov I. V., “Processing of the information from side-scan sonar”, *Proceedings of SPIE*, **10035**, (2016), 100352C.

[7] Kovalenko E. O., Sushchenko A. A., Kan V. A., “Focusing of sonar images as an inverse problem for radiative transfer equation”, *Proceedings of SPIE*, **10833**, (2018), 108336D.

[8] Prokhorov I. V., Sushchenko A. A., Kim A., “Initial boundary value problem for the radia-

tive transfer equation with diffusion matching conditions", *Journal of Applied and Industrial Mathematics*, **11**:1, (2017), 115–124.

[9] Prokhorov I. V., Sushchenko A. A., "The Cauchy problem for the radiatve transfer equation in an unbounded medium", *Dal'nevost. Mat. Zh.*, **18**:1, (2018), 101–111.

[10] Prokhorov I. V., Kovalenko E. O., "Determination of the bottom scattering coefficient in multi-beam probing of the ocean", *Far Eastern Mathematical Journal*, **19**:2, (2019), 206–222.

[11] Amosov A. A., "Initial-Boundary Value Problem for the Non-Stationary Radiative Transfer Equation with Fresnel Reflection and Refraction Conditions", *Journal of Mathematical Sciences*, **231**:3, (2018), 279–337.

[12] Amosov A. A., "Initial-Boundary Value Problem for the Non-stationary Radiative Transfer Equation with Diffuse Reflection and Refraction Conditions", *Journal of Mathematical Sciences*, **235**:2, (2018), 117–137.

[13] Amosov A. A., "Nonstationary radiation transfer through a multilayered medium with reflection and refraction conditions", *Mathematical Methods in the Applied Sciences*, **41**:17, (2018), 8115–8135.

[14] Prokhorov I. V., "The Cauchy Problem for the Radiation Transfer Equation with Fresnel and Lambert Matching Conditions", *Math. Notes*, **105**:1, (2019), 80–90.

[15] Kim A., Prokhorov I. V., "Initial-boundary value problem for a radiative transfer equation with generalized matching conditions", *Siberian Electronic Mathematical Reports*, **16**, (2019), 1036–1056.

Received by the editors
March 27, 2023

This work was supported by the Ministry of Science and Higher Education, (agreement no. 075-02-2023-946) within the research and development task no. AAAA-A20-120120390006-0.

Коваленко Е. О., Сущенко А. А. Фокусировка гидроакустических изображений по данным многоракурсного зондирования. *Дальневосточный математический журнал.* 2024. Т. 24. № 2. С. 193–199.

АННОТАЦИЯ

Исследуется обратная задача для нестационарного уравнения переноса излучения. Искомой функцией является коэффициент донного рассеяния, который содержится в граничных условиях задачи. Источник предполагается импульсным, а приемник имеет диаграмму направленности приемной антенны конечной ширины, что влияет на расфокусировку объектов при построении гидроакустического изображения. Для решения задачи авторами предложен алгоритм многоракурсной фокусировки, который был опробован на данных, полученных на основе реального изображения морского дна. Проведен анализ точности решения обратной задачи в зависимости от ширины диаграммы направленности приемной антенны.

Ключевые слова: *уравнение переноса излучения, обратная задача, донное и объемное рассеяние, многолучевое зондирование.*