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Structure of essential spectrum and discrete spectra of the energy operator of five-electron systems in the Hubbard model. Fourth quartet state

We consider the energy operator of five-electron systems in the Hubbard model and investigate the structure of essential spectra and discrete spectrum of the system in the fourth quartet state of the system. We show that the essential spectrum of the system in fourth quartet states is the union of at most seven segments, and discrete spectrum of the system is at most one point.

Key words: *five-electron system, Hubbard Model, quartet state, doublet state, sextet state, essential spectra, discrete spectra.*

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Introduction

In the early 1970s, three papers [1–3], where a simple model of a metal was proposed that has become a fundamental model in the theory of strongly correlated electron systems, appeared almost simultaneously and independently. In that model, a single nondegenerate electron band with a local Coulomb interaction is considered. The model Hamiltonian contains only two parameters: the matrix element t of electron hopping from a lattice site to a neighboring site and the parameter U of the one-site Coulomb repulsion of two-electrons. In the secondary quantization representation, the Hamiltonian can be written as

$$H = t\sum_{m,\tau,\gamma} a^{+}_{m,\gamma}a_{m+\tau,\gamma} + U\sum_{m} a^{+}_{m,\uparrow}a_{m,\uparrow}a^{+}_{m,\downarrow}a_{m,\downarrow},$$

where $a^{+}_{m,\gamma}$ and $a_{m,\gamma}$ denote Fermi operators of creation and annihilation of an electron with spin γ on a site m and the summation over τ means summation over the nearest neighbors on the lattice.

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The model proposed in [1–3] was called the Hubbard model after John Hubbard, who made a fundamental contribution to studying the statistical mechanics of that system, although the local form of Coulomb interaction was first introduced for an impurity model in a metal by Anderson [4]. We also recall that the Hubbard model is a particular case of the Shubin-Wonsowsky polaron model [5], which had appeared 30 years before [1–3]. In the Shubin-Wonsowsky model, along with the on-site Coulomb interaction, the interaction of electrons on neighboring sites is also taken into account.

The Hubbard model is an approximation used in solid state physics to describe the transition between conducting and insulating states. It is the simplest model describing particle interaction on a lattice. Particles can be fermions, as in Hubbard's original work, and also bosons. The simplicity and sufficiency of Hubbard Hamiltonian have made the Hubbard model very popular and effective for describing strongly correlated electron systems.

The Hubbard model well describes the behavior of particles in a periodic potential at sufficiently low temperatures such that all particles are in the lower Bloch band and long-range interactions can be neglected. If the interaction between particles at different sites is taken into account, then the model is often called the extended Hubbard model. It was proposed for describing electrons in solids, and it remains especially interesting since then for studying high-temperature superconductivity. Later, the extended Hubbard model also found applications in describing the behavior of ultracold atoms in optical lattices.

In considering electrons in solids, the Hubbard model can be considered a sophisticated version of the model of strongly bound electrons, involving only the electron hopping term in the Hamiltonian. In the case of strong interactions, these two models can give essentially different results. The Hubbard model exactly predicts the existence of so-called Mott insulators, where conductance is absent due to strong repulsion between particles.

The Hubbard model is based on the approximation of strongly coupled electrons. In the strongcoupling approximation, electrons initially occupy orbital's in atoms (lattice sites) and then hop over to other atoms, thus conducting the current. Mathematically, this is represented by the so-called hopping integral. This process can be considered the physical phenomenon underlying the occurrence of electron bands in crystal materials. But the interaction between electrons is not considered in more general band theories. In addition to the hopping integral, which explains the conductance of the material, the Hubbard model contains the so-called on-site repulsion, corresponding to the Coulomb repulsion between electrons. This leads to a competition between the hopping integral, which depends on the mutual position of lattice sites, and the on-site repulsion, which is independent of the atom positions. As a result, the Hubbard model explains the metal–insulator transition in oxides of some transition metals. When such a material is heated, the distance between nearest-neighbor sites increases, the hopping integral decreases, and on-site repulsion becomes dominant.

The Hubbard model is currently one of the most extensively studied multielectron models of metals [6–10]. But little is known about exact results for the spectrum and wave functions of the crystal described by the Hubbard model, and obtaining the corresponding statements is therefore of great interest. The spectrum and wave functions of the system of two electrons in a crystal described by the Hubbard Hamiltonian were studied in [7]. It is known that two-electron systems can be in two states, triplet and singlet [6–10]. It was proved in [7] that the spectrum of the system Hamiltonian H^t in the triplet state is purely continuous and coincides with a segment $[m, M]$, and the operator H^s of the system in the singlet state, in addition to the continuous spectrum $[m, M]$, has a unique antibound state for some values of the quasimomentum. For the antibound state, correlated motion of the electrons is realized under which the contribution of binary states is large. Because the system is closed, the energy must remain constant and large. This prevents the electrons from being separated by long distances. Next, an essential point is that bound states (sometimes called scattering-type states) do not form below the continuous spectrum. This can be easily understood because the interaction is repulsive. We note that a converse situation is realized for $U < 0$: below the continuous spectrum, there is a bound state (antibound states are absent) because the electrons are then attracted to one another.

For the first band, the spectrum is independent of the parameter U of the on-site Coulomb interaction of two electrons and corresponds to the energy of two noninteracting electrons, being exactly equal to the triplet band. The second band is determined by Coulomb interaction to a much greater degree: both the amplitudes and the energy of two electrons depend on U, and the band itself disappears as $U \to 0$ and increases without bound as $U \to \infty$. The second band largely corresponds to a one-particle state, namely, the motion of the doublet, i.e., two-electron bound states.

The spectrum and wave functions of the system of three electrons in a crystal described by the Hubbard Hamiltonian were studied in [11]. In the three-electron systems are exists quartet state, and two type doublet states. In the work [11] proved that the essential spectrum of the system in a quartet state consists of a single segment and the three-electron bound state is absent. It is also shown that the essential spectrum of the system in doublet states is the union of at most three segments, and it is proved that three-electron bound states exist in doublet states. In addition, the spectra of this doublet states are the different.

The spectrum and wave functions of the system of four electrons in a crystal described by the Hubbard Hamiltonian were studied in [12, 13]. In the four-electron systems are exists a six states: quintet state, three type triplet state, and two type singlet states. In the work [12] investigated the spectrum and wave functions of four-electron systems in a Hubbard model in triplet states. In the work [13] considered the spectrum and wave functions of four-electron systems in a Hubbard model in a quintet and singlet states.

1 Energy operator of five-electron systems in the Hubbard Model. Fourth quartet state

Here, we consider the energy operator of five-electron systems in the Hubbard model and investigate the structure of the essential spectrum and discrete spectra of the system for fourth quartet state. The Hamiltonian of the considered model has the form

$$H = A\sum_{m,\gamma} a^+_{m,\gamma} a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^+_{m,\gamma} a_{m+\tau,\gamma} + U\sum_{m} a^+_{m,\uparrow} a_{m,\uparrow} a^+_{m,\downarrow} a_{m,\downarrow}. \tag{1}$$

Here, A is the electron energy at a lattice site, B is the transfer integral between neighboring sites (we assume that $B > 0$ for convenience), $\tau = \pm e_j, j = 1, 2, ..., \nu$, where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U is the parameter of the on-site Coulomb interaction of two electrons, γ is the spin index,$\gamma =\uparrow$ or $\gamma =\downarrow$, $\uparrow$ or $\downarrow$ denote the spin values $\frac{1}{2}$ or $-\frac{1}{2}$, and $a^+_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^\nu$.

The energy of the system depends on its total spin S. Along with the Hamiltonian, the N_e electron system is characterized by the total spin S, $S = S_{max}, S_{max} - 1, \dots, S_{min}, S_{max} = \frac{N_e}{2}, S_{min} = 0, \frac{1}{2}$. Hamiltonian (1) commutes with all components of the total spin operator $S = (S^+, S^-, S^z)$, and the structure of eigenfunctions and eigenvalues of the system therefore depends on S.

The Hamiltonian H acts in the antisymmetric Fock space $\widetilde{\mathcal{H}}_{as}$. Below we give the constructions of the Fock space $\mathcal{F}(\mathcal{H})$.

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{H}^n$ the $n-$ fold tensor product $\mathcal{H}^n = \mathcal{H} \bigotimes \mathcal{H} \bigotimes ... \bigotimes \mathcal{H}$. We set $\mathcal{H}^0 = C$ and $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^\infty \mathcal{H}^n$. The $\mathcal{F}(\mathcal{H})$ is called the Fock space over $\mathcal{H}$; it will be separably, if $\mathcal{H}$ is. For example, if $\mathcal{H} = L_2(R)$, then an element $\psi \in \mathcal{F}(\mathcal{H})$ is a sequence of functions

$$\psi = \Big\{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_3(x_1, x_2, x_3), \dots\Big\},$$

so that

$$\big|\psi_0\big|^2 + \sum_{n=1}^{\infty} \int_{R^n} \big|\psi_n(x_1, x_2, \dots, x_n)\big|^2 dx_1 dx_2 \dots dx_n < \infty.$$

Actually, it is not $\mathcal{F}(\mathcal{H})$, itself, but two of its subspaces which are used most frequently in quantum field theory. These two subspaces are constructed as follows: Let $\mathcal{P}_n$ be the permutation group on n elements, and let $\{\psi_n\}$ be a basis for space $\mathcal{H}$. For each $\sigma \in \mathcal{P}_n$, we define an operator (which we also denote by σ) on basis elements $\mathcal{H}^n$, by

$$\sigma\left(\varphi_{k_1} \bigotimes \varphi_{k_2} \bigotimes \cdots \bigotimes \varphi_{k_n}\right) = \varphi_{k_{\sigma(1)}} \bigotimes \varphi_{k_{\sigma(2)}} \bigotimes \cdots \bigotimes \varphi_{k_{\sigma(n)}}.$$

The operator σ extends by linearity to a bounded operator (of norm one) on space $\mathcal{H}^n$, so we can define $S_n = \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} \sigma$. That the operator S_n is the operator of orthogonal projection: $S_n^2 = S_n$, and $S_n^* = S_n$. The range of S_n is called $n-$ fold symmetric tensor product of $\mathcal{H}$. In the case, where $\mathcal{H} = L_2(R)$ and

$$\mathcal{H}^n = L_2(R) \bigotimes L_2(R) \bigotimes \cdots \bigotimes L_2(R) = L_2(R^n),$$

$S_n\mathcal{H}^n$ is just the subspace of $L_2(R^n)$, of all functions, left invariant under any permutation of the variables. We now define $\mathcal{F}_s(\mathcal{H}) = \bigoplus_{n=0}^\infty S_n\mathcal{H}^n$. The space $\mathcal{F}_s(\mathcal{H})$ is called the symmetrical Fock space over $\mathcal{H}$, or Boson Fock space over $\mathcal{H}$.

Let $\varepsilon(.)$ is function from $\mathcal{P}_n$ to $\{1,-1\}$, which is one on even permutations and minus one on odd permutations. Define $A_n = \frac{1}{n!}\sum\limits_{\sigma\in\mathcal{P}_n}\varepsilon(\sigma)\sigma$; then A_n is an orthogonal projector on $\mathcal{H}^n$. $A_n\mathcal{H}^n$ is called the $n-$ fold antisymmetrical tensor product of $\mathcal{H}$. In the case where $\mathcal{H}=L_2(R)$, $A_n\mathcal{H}^n$ is just the subspace of $L_2(R^n)$, consisting of those functions odd under interchange of two coordinates. The subspace $\mathcal{F}_a(\mathcal{H})=\bigoplus_{n=0}^{\infty}A_n\mathcal{H}^n$ is called the antisymmetrical Fock space over $\mathcal{H}$, or the Fermion Fock space over $\mathcal{H}$.

Let φ_0 be the vacuum vector in the space $\widetilde{\mathcal{H}}_{as}$.

The fourth quartet state corresponds to the free motion of five electrons over the lattice, and their interactions with the basis functions:

$$^4q^{3/2}_{m,n,r,t,l\in Z^\nu}=a^+_{m,\uparrow}a^+_{n,\uparrow}a^+_{r,\uparrow}a^+_{t,\downarrow}a^+_{l,\uparrow}\varphi_0.$$

The subspace $^4\widetilde{\mathcal{H}}^q_{3/2}$, corresponding to the fourth quartet state is the set of all vectors of the form

$$\psi=\sum_{m,n,r,t,l\in Z^\nu}\widetilde{f}(m,n,r,t,l)^4q^{3/2}_{m,n,r,t,l},\quad \widetilde{f}\in l_2^{as},$$

where l_2^{as} is the subspace of antisymmetric functions in the space $l_2((Z^\nu)^5)$. We denote by $^4H^q_{3/2}$ the restriction of the operator H to the space $^4\widetilde{\mathcal{H}}^q_{3/2}$. We call the operator $^4H^q_{3/2}$ the five-electron fourth quartet state operator.

Theorem 1. *The subspace $^4\widetilde{\mathcal{H}}^q_{3/2}$ is invariant under the operator H, and the operator $^4H^q_{3/2}$ is a bounded self-adjoint operator. It generates a bounded self-adjoint operator $^4\overline{H}^q_{3/2}$, acting in the space l_2^{as} as*

$$^4\overline{H}^q_{3/2}\psi=$$
$$=5Af(m,n,r,t,l)+B\sum_\tau\big[f(m+\tau,n,r,t,l)+f(m,n+\tau,r,t,l)+f(m,n,r+\tau,t,l)+ \quad (2)$$
$$+f(m,n,r,t+\tau,l)+f(m,n,r,t,l+\tau)\big]+U(\delta_{m,t}+\delta_{n,t}+\delta_{r,t}+\delta_{l,t})f(m,n,r,t,l),$$

where $\delta_{k,j}$ is the Kronecker symbol. The operator $^4H^q_{3/2}$, acts on a vector $\psi\in{}^4\widetilde{\mathcal{H}}^q_{3/2}$ as

$$^4H^q_{3/2}\psi=\sum_{m,n,r,t,l}(^4\overline{H}^q_{3/2}f)(m,n,r,t,l)^4q^{3/2}_{m,n,r,t,l}.$$

P r o o f. We act with the Hamiltonian H on vectors $\psi\in{}^4\widetilde{\mathcal{H}}^q_{3/2}$ using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, $\{a_{m,\gamma},a^+_{n,\beta}\}=\delta_{m,n}\delta_{\gamma,\beta}$, $\{a_{m,\gamma},a_{n,\beta}\}=\{a^+_{m,\gamma},a^+_{n,\beta}\}=\theta$, and also take into account that $a_{m,\gamma}\varphi_0=\theta$, where θ is the zero element of $^4\widetilde{\mathcal{H}}^q_{3/2}$. This yields the statement of the theorem. □

Lemma 1. *The spectra of the operators $^4H^q_{3/2}$ and $^4\overline{H}^d_{3/2}$ coincide.*

P r o o f. Because $^4H^q_{3/2}$ and $^4\overline{H}^d_{3/2}$ are bounded self-adjoint operators, it follows that if $\lambda\in\sigma(^4H^q_{3/2})$, then the Weyl criterion (see [14], chapter VII, paragraph 3, pp. 262- 263) implies that there is a sequence $\{\psi_i\}_{i=1}^{\infty}$ such that $\psi_i = \sum\limits_{m,n,r,t,l} f_i(m,n,r,t,l)^4 q_{m,n,r,t,l}^{3/2}$, $||\psi_i|| = 1$, and

$$\lim_{i\to\infty} ||({}^4H^q_{3/2} - \lambda)\psi_i|| = 0. \tag{3}$$

On the other hand,

$$\left\| \left({}^4H^q_{3/2} - \lambda\right)\psi_i \right\|^2 = \left(\left({}^4H^q_{3/2} - \lambda\right)\psi_i, \left({}^4H^q_{3/2} - \lambda\right)\psi_i \right) =$$
$$= \sum_{m,n,r,t,l} \left\| \left({}^4\overline{H}^q_{3/2} - \lambda\right) f_i(m,n,r,t,l) \right\|^2 \left(a^+_{m,\uparrow} a^+_{n,\uparrow} a^+_{r,\uparrow} a^+_{t,\downarrow} a^+_{l,\uparrow}\varphi_0, a^+_{m,\uparrow} a^+_{n,\uparrow} a^+_{r,\uparrow} a^+_{t,\downarrow} a^+_{l,\uparrow}\varphi_0 \right) =$$
$$= \sum_{m,n,r,t,l} \left\| \left({}^4\overline{H}^q_{3/2} - \lambda\right) f_i(m,n,r,t,l) \right\|^2 \left(a_{l,\uparrow} a_{t,\downarrow} a_{r,\uparrow} a_{n,\uparrow} a_{m,\uparrow} a^+_{m,\uparrow} a^+_{n,\uparrow} a^+_{r,\uparrow} a^+_{t,\downarrow} a^+_{l,\uparrow}\varphi_0, \varphi_0 \right) =$$
$$= \sum_{m,n,r,t,l} \left\| \left({}^4\overline{H}^q_{3/2} - \lambda\right) f_i(m,n,r,t,l) \right\|^2 (\varphi_0, \varphi_0) = \left\| \left({}^4\overline{H}^q_{3/2} - \lambda\right) F_i \right\|^2,$$

and

$$||F_i||^2 = \sum_{m,n,r,t,l} |f_i(m,n,r,t,l)|^2 = ||\psi_i||^2 = 1.$$

From this and formulas (2), we find that $||{}^4\overline{H}^q_{3/2}F_i - \lambda F_i|| \to 0$, as $i \to \infty$, and $F_i = \sum\limits_{m,n,r,t,l} f_i(m,n,r,t,l)$. This implies that $\lambda \in \sigma({}^4\overline{H}^q_{3/2})$. Consequently, $\sigma({}^4H^q_{3/2}) \subset \sigma({}^4\overline{H}^q_{3/2})$.

Conversely, let $\overline{\lambda} \in \sigma({}^4\overline{H}^q_{3/2})$. Again by the Weyl criterion, there then exists a sequence $\{F_i\}_{i=1}^{\infty}$ such that $||F_i|| = 1$ and $\lim\limits_{i\to\infty} ||({}^4\overline{H}^q_{3/2} - \overline{\lambda})\psi_i|| = 0$.

Setting $F_i = \sum\limits_{m,n,r,t,l} f_i(m,n,r,t,l)$, we have $||F_i|| = \left(\sum\limits_{m,n,r,t,l} |f_i(m,n,r,t,l)|^2 \right)^{\frac{1}{2}}$, we conclude that $||\psi_i|| = ||F_i|| = 1$ and $||({}^4\overline{H}^q_{3/2} - \overline{\lambda})F_i|| = ||({}^4\overline{H}^q_{3/2} - \overline{\lambda})\psi_i|| \to 0$ as $i \to \infty$. This means that $\overline{\lambda} \in \sigma({}^4H^q_{3/2})$ and hence $\sigma({}^4\overline{H}^q_{3/2}) \subset \sigma({}^4H^q_{3/2})$. These two relations imply $\sigma({}^4H^q_{3/2}) = \sigma({}^4\overline{H}^q_{3/2})$. □

We let $\mathcal{F}$ denote the Fourier transform:

$$\mathcal{F} : l_2((Z^\nu)^5) \to L_2((T^\nu)^5) \equiv {}^4\widetilde{\mathcal{H}}^q_{3/2},$$

where T^ν is the $\nu-$ dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, $\lambda(T^\nu) = 1$.

We set ${}^4\widetilde{H}^q_{3/2} = \mathcal{F}\, {}^4\overline{H}^q_{3/2}\mathcal{F}^{-1}$. In the quasimomentum representation, the operator ${}^4\overline{H}^q_{3/2}$ acts in the Hilbert space $L_2^{as}((T^\nu)^5)$, where L_2^{as} is the subspace of antisymmetric functions in $L_2((T^\nu)^5)$.

Theorem 2. *The Fourier transform of operator* ${}^4\overline{H}^q_{3/2}$ *is an bounded self-adjoint oper-*

ator ${}^4\widetilde{H}^q_{3/2} = \mathcal{F}\,{}^4\overline{H}^q_{3/2}\,\mathcal{F}^{-1}$, *acting in the space* ${}^4\widetilde{\mathcal{H}}^q_{3/2}$ *by the formula*

$$\left({}^4\widetilde{H}^q_{3/2}\widetilde{f}\right)(\lambda,\mu,\gamma,\theta,\eta)=\left\{5A+2B\sum_{i=1}^{\nu}\left[\cos\lambda_i+\cos\mu_i+\cos\gamma_i+\cos\theta_i+\cos\eta_i\right]\right\}\times$$
$$\times\widetilde{f}(\lambda,\mu,\gamma,\theta,\eta)+U\int_{T^\nu}\widetilde{f}(s,\mu,\gamma,\lambda+\theta-s,\eta)ds+U\int_{T^\nu}\widetilde{f}(\lambda,s,\gamma,\mu+\theta-s,\eta)ds+ \quad (4)$$
$$+U\int_{T^\nu}\widetilde{f}(\lambda,\mu,s,\gamma+\theta-s,\eta)ds+U\int_{T^\nu}\widetilde{f}(\lambda,\mu,\gamma,s,\theta+\eta-s)ds.$$

The prove Theorem 2, the Fourier transform of (2) should be considered directly.

Using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces [14], we can verify that the operator ${}^4\widetilde{H}^q_{3/2}$ can be represented in the form

$${}^4\widetilde{H}^q_{3/2}=\widetilde{H}^1_2\bigotimes I\bigotimes I+I\bigotimes\widetilde{H}^2_2\bigotimes I+I\bigotimes I\bigotimes\widetilde{H}^3_2, \quad (5)$$

where

$$\left(\widetilde{H}^1_2\widetilde{f}\right)(\lambda,\gamma)=-\left\{2A+2B\sum_{i=1}^{\nu}\left[\cos\lambda_i+\cos\gamma_i\right]\right\}\widetilde{f}(\lambda,\gamma)+U\int_{T^\nu}\widetilde{f}(s,\lambda+\theta-s)ds,$$
$$\left(\widetilde{H}^2_2\widetilde{f}\right)(\mu,\theta)=\left\{A+2B\sum_{i=1}^{\nu}\cos\mu_i\right\}\widetilde{f}(\mu,\theta)-U\int_{T^\nu}\widetilde{f}(s,\mu+\theta-s)ds,$$

and

$$\left(\widetilde{H}^3_2\widetilde{f}\right)(\theta,\eta)=\left\{2A+2B\sum_{i=1}^{\nu}\left[\cos\theta_i+\cos\eta_i\right]\right\}\widetilde{f}(\theta,\eta)+U\int_{T^\nu}\widetilde{f}(s,\theta+\eta-s)ds+U\int_{T^\nu}\widetilde{f}(s,\gamma+\eta-s)ds,$$

and I is the unit operator in space of two-electron states $\widetilde{\mathcal{H}}_2$.

Therefore, we must investigate the spectrum of the operators $\widetilde{H}^1_2$, $\widetilde{H}^2_2$, and $\widetilde{H}^3_2$.

Let the total quasimomentum of the system $\Lambda_1=\lambda+\gamma$ be fixed. We let $L_2(\Gamma_{\Lambda_1})$ denote the space of functions that are square integrable on the manifold $\Gamma_{\Lambda_1}=\{(\lambda,\gamma): \lambda+\gamma=\Lambda_1\}$. It is known [15] that the operator $\widetilde{H}^1_2$ and the space $\widetilde{\mathcal{H}}^{as}_2\equiv L^{as}_2((T^\nu)^2)$, where $L^{as}_2((T^\nu)^2)$ is the subspace of antisymmetric functions in $L_2((T^\nu)^2)$, can be decomposed into a direct integrals

$$\widetilde{H}^1_2=\int_{T^\nu}\bigoplus\widetilde{H}^1_{2\Lambda_1}d\Lambda_1,\qquad \widetilde{\mathcal{H}}^{as}_2=\int_{T^\nu}\bigoplus\widetilde{\mathcal{H}}^{as}_{2\Lambda_1}d\Lambda_1$$

of operators $\widetilde{H}^1_{2\Lambda_1}$ and spaces $\widetilde{\mathcal{H}}^{as}_{2\Lambda_1}$ such that the spaces $\widetilde{\mathcal{H}}^{as}_{2\Lambda_1}$ are invariant under the operators $\widetilde{H}^1_{2\Lambda_1}$, and each operator $\widetilde{H}^1_{2\Lambda_1}$ acts in space $\widetilde{\mathcal{H}}^{as}_{2\Lambda_1}$ as

$$\left(\widetilde{H}^1_{2\Lambda_1}f_{\Lambda_1}\right)(\lambda)=-\left\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda^i_1}{2}\cos\left(\frac{\Lambda^i_1}{2}-\lambda\right)\right\}f_{\Lambda_1}(\lambda)+U\int_{T^\nu}f_{\Lambda_1}(s)ds,$$

here $f_{\Lambda_1}(s) = f(s, \Lambda_1 - s)$.

It is known that the continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$ is independent of the parameter U and consists of the segments

$$G^\nu_{\Lambda_1} = \left[m^\nu_{\Lambda_1}, M^\nu_{\Lambda_1}\right] = \left[-2A - 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_1^i}{2}, -2A + 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_1^i}{2}\right].$$

Definition 1. *The eigenfunction* $\varphi_{\Lambda_1} \in L_2^{as}((T^\nu)^2)$ *of the operator* $\widetilde{H}^1_{2\Lambda_1}$ *corresponding to an eigenvalue* $z_{\Lambda_1} \notin G^\nu_{\Lambda_1}$ *is called a at* $U > 0$ *antibound state (at* $U < 0$ *bound state) of operator* $\widetilde{H}^1_{2\Lambda_1}$ *with the quasimomentum* Λ_1*, and the quantity* z_{Λ_1} *is called the energy of this state.*

We set

$$\Delta_\nu(z) = 1 + U\int\limits_{T^\nu} \frac{ds_1 ds_2 \dots ds_\nu}{-2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos(\frac{\Lambda_1^i}{2} - s_i) - z}.$$

Lemma 2. *A number* $z = z_0 \notin \sigma_{cont}(\widetilde{H}^1_{2\Lambda_1})$ *is an eigenvalue of operator* $\widetilde{H}^1_{2\Lambda_1}$ *if and only if it is a zero of the function* $\Delta_\nu(z)$*, i.e.,* $\Delta_\nu(z_0) = 0$.

P r o o f. Let the number $z = z_0 \notin [m^\nu_{\Lambda_1}, M^\nu_{\Lambda_1}]$ be an eigenvalue of the operator $\widetilde{H}^1_{2\Lambda_1}$, and $\varphi_{\Lambda_1}(x)$ be the corresponding eigenfunction, i.e.

$$-\left\{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2} - \lambda\right)\right\}\varphi_{\Lambda_1}(\lambda) + U\int\limits_{T^\nu}\varphi_{\Lambda_1}(s)ds = z_0\varphi_{\Lambda_1}(x).$$

Let

$$\psi_{\Lambda_1}(x) = \left[-\left[2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2} - \lambda_i\right)\right] - z\right]\varphi_{\Lambda_1}(x).$$

Then

$$\psi_{\Lambda_1}(x) + \int\limits_{T^\nu}\frac{U}{-\left[2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2} - t_i\right)\right] - z}\psi_{\Lambda_1}(t)dt = 0,$$

i.e. the number $\mu = 1$ is a eigenvalue of the operator $K_{\Lambda_1}(z)$, where

$$\Big(K_{\Lambda_1}(z)f_{\Lambda_1}\Big)(x) = \int\limits_{T^\nu}\frac{U}{-\left[2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2} - t_i\right)\right] - z}f_{\Lambda_1}(t)dt.$$

It then follows that $\Delta_\nu(z_0) = 0$.

Now let $z = z_0$ a zero of the function $\Delta_\nu(z)$, i.e. $\Delta_\nu(z_0) = 0$. It follows from the Fredholm theorem that the homogeneous equation

$$\psi_{\Lambda_1}(x) + U\int\limits_{T^\nu}\frac{\psi_{\Lambda_1}(s)ds_1ds_2...ds_\nu}{-2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2} - s_i\right) - z} = 0$$

has a nontrivial solution. This means that the number $z = z_0$ is an eigenvalue of the operator $\widetilde{H}^1_{2\Lambda_1}$. □

We consider the one-dimensional case.

Theorem 3. *At $\nu = 1$ and $U < 0$ ($U > 0$), and for all values of parameters of the Hamiltonian, the operator $\widetilde{H}^1_{2\Lambda_1}$ has a unique two-electron bound state (antibound state) φ with the energy value $z_1 = -2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$ $\left(\widetilde{z}_1 = -2A + \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}\right)$, that is below (above) the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, i.e., $z_1 < m^1_{\Lambda_1}$ $\left(\widetilde{z}_1 > M^1_{\Lambda_1}\right)$.*

P r o o f. Let $\nu = 1$ and $U < 0$. Then the continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$ consists in segment $\left[-2A - 4B\cos\frac{\Lambda_1}{2}, -2A + 4B\cos\frac{\Lambda_1}{2}\right]$. In the one-dimensional case, if $U < 0$, then the function $\Delta_\nu(z)$ are monotonically decreasing function of z the outside of continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$, i.e., in $(-\infty, m^1_{\Lambda_1})$ and in $(M^1_{\Lambda_1}, +\infty)$. For $z < m^1_{\Lambda_1}$ the function $\Delta_1(z)$ decreasing from 1 to $-\infty$, $\Delta_1(z) \to 1$ as $z \to -\infty$, $\Delta_1(z) \to -\infty$ as $z \to m^1_{\Lambda_1} - 0$. Therefore, below the value $m^1_{\Lambda_1}$ the function $\Delta_\nu(z)$ has a single zero at the point $z_1 = -2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$. For $z > M^1_{\Lambda_1}$ the function $\Delta_1(z)$ decreasing from $+\infty$ to 1, $\Delta_1(z) \to +\infty$ as $z \to M^1_{\Lambda_1} + 0$, $\Delta_1(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^1_{\Lambda_1}$ the function $\Delta_\nu(z)$ cannot vanish.

If $\nu = 1$ and $U > 0$, then the function $\Delta_\nu(z)$ increases monotonically outside the continuous spectrum domain of the operator $\widetilde{H}^1_{2\Lambda_1}$. For $z < m^1_{\Lambda_1}$ the function $\Delta_1(z)$ increases from 1 to $+\infty$, $\Delta_1(z) \to 1$ as $z \to -\infty$, $\Delta_1(z) \to +\infty$ as $z \to m^1_{\Lambda_1} - 0$. Therefore, below the value $m^1_{\Lambda_1}$ the function $\Delta_\nu(z)$ cannot vanish. For $z > M^1_{\Lambda_1}$ and $U > 0$, the function $\Delta_1(z)$ increases from $-\infty$ to 1, $\Delta_1(z) \to -\infty$ as $z \to M^1_{\Lambda_1} + 0$, $\Delta_1(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^1_{\Lambda_1}$ the function $\Delta_\nu(z)$ has a single zero at the point $\widetilde{z}_1 = -2A + \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$. □

In the two-dimensional case, we have similar results.

Theorem 4. *At $\nu = 2$ and $U < 0$ ($U > 0$), and for all values of parameters of the Hamiltonian, the operator $\widetilde{H}^1_{2\Lambda_1}$ has a unique two-electron bound state (antibound state) φ with the energy value $\widehat{z}_1$ (z_1'), that is below (above) the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, i.e., $\widehat{z}_1 < m^2_{\Lambda_1}$ ($z_1' > M^2_{\Lambda_1}$).*

P r o o f. Let $\nu = 2$ and $U < 0$. Then the continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$ consists in segment $\left[-2A - 4B\sum\limits_{i=1}^{2}\cos\frac{\Lambda_1^i}{2}, -2A + 4B\sum\limits_{i=1}^{2}\cos\frac{\Lambda_1^i}{2}\right]$. In the two-dimensional case, if $U < 0$, then the function $\Delta_\nu(z)$ are monotonically decreasing function of z the outside of continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$, i.e., in $(-\infty, m^2_{\Lambda_1})$ and in $(M^2_{\Lambda_1}, +\infty)$. For $z < m^2_{\Lambda_1}$ the function $\Delta_2(z)$ decreasing from 1 to $-\infty$, $\Delta_2(z) \to 1$ as $z \to -\infty$, $\Delta_2(z) \to -\infty$ as $z \to m^2_{\Lambda_1} - 0$. Therefore, below the value $m^2_{\Lambda_1}$ the function $\Delta_\nu(z)$ has a single zero at the point $\widehat{z}_1$. For $z > M^2_{\Lambda_1}$ the function $\Delta_2(z)$ decreasing from $+\infty$ to 1, $\Delta_2(z) \to +\infty$ as $z \to M^2_{\Lambda_1} + 0$, $\Delta_2(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^2_{\Lambda_1}$ the function $\Delta_\nu(z)$ cannot vanish.

If $\nu = 2$ and $U > 0$, then the function $\Delta_\nu(z)$ increases monotonically outside the continuous spectrum domain of operator $\widetilde{H}^1_{2\Lambda_1}$. For $z < m^2_{\Lambda_1}$ the function $\Delta_2(z)$ increases from 1 to $+\infty$, $\Delta_2(z) \to 1$ as $z \to -\infty$, $\Delta_2(z) \to +\infty$ as $z \to m^2_{\Lambda_1} - 0$. Therefore, below the value $m^2_{\Lambda_1}$ the function $\Delta_\nu(z)$ cannot vanish. For $z > M^2_{\Lambda_1}$ the function $\Delta_2(z)$

increases from $-\infty$ to 1, $\Delta_2(z) \to -\infty$ as $z \to M^2_{\Lambda_1}+0$, $\Delta_2(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^2_{\Lambda_1}$ the function $\Delta_\nu(z)$ has a single zero at the point $z_1^{'}$. □

We now consider three-dimensional case. Here and hereafter, we denote

$$M = \int\limits_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^3 \cos\frac{\Lambda_1^i}{2}\left(1-\cos\left(\frac{\Lambda_1^i}{2}-s_i\right)\right)} \quad \text{and} \quad m = \int\limits_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^3 \cos\frac{\Lambda_1^i}{2}\left(1+\cos\left(\frac{\Lambda_1^i}{2}-s_i\right)\right)}.$$

Let $\nu = 3$ and $U < 0$.

Theorem 5. *a). If $U<0$ and $U<-\frac{4B}{M}$, then the operator $\widetilde{H}^1_{2\Lambda_1}$ has a unique bound state φ with the energy value $\widetilde{\widetilde{z}}_1$, that is below the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, i.e., $\widetilde{\widetilde{z}}_1 < m^3_{\Lambda_1}$.*

b). If $U<0$ and $-\frac{4B}{M} \le U < 0$, then the operator $\widetilde{H}^1_{2\Lambda_1}$ has no bound state with the energy value, that is below the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$.

c). If $U>0$ and $U>\frac{4B}{m}$, then the operator $\widetilde{H}^1_{2\Lambda_1}$ has a unique bound state φ with the energy value $z_1^{''}$, that is above the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, i.e., $z_1^{''} > M^3_{\Lambda_1}$.

d). If $U>0$ and $0<U\le\frac{4B}{m}$, then the operator $\widetilde{H}^1_{2\Lambda_1}$ has no bound state with the energy value, that is above the continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$.

P r o o f. Let $\nu = 3$ and $U<0$. Then the continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$ consists of segment $\left[-2A-4B\sum\limits_{i=1}^3\cos\frac{\Lambda_1^i}{2}, -2A+4B\sum\limits_{i=1}^3\cos\frac{\Lambda_1^i}{2}\right]$. In the three-dimensional case, if $U<0$, then the function $\Delta_\nu(z)$ are monotonically decreasing function of z the outside of continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, i.e., in $(-\infty, m^3_{\Lambda_1})$ and in $(M^3_{\Lambda_1}, +\infty)$. For $z<m^3_{\Lambda_1}$ the function $\Delta_3(z)$ decreasing from 1 to $1+\frac{UM}{4B}$, $\Delta_3(z)\to 1$ as $z\to-\infty$, $\Delta_3(z) \to 1+\frac{UM}{4B}$ as $z \to m^3_{\Lambda_1}-0$. Therefore, the below of values $m^3_{\Lambda_1}$ the function $\Delta_\nu(z)$ has a single zero at the point $\widetilde{\widetilde{z}}_1$, if $1+\frac{UM}{4B}<0$, i.e., $U<-\frac{4B}{M}$. For $z>M^3_{\Lambda_1}$ the function $\Delta_3(z)$ decreasing from $1-\frac{Um}{4B}>1$ to 1, $\Delta_3(z)\to 1-\frac{Um}{4B}$ as $z\to M^3_{\Lambda_1}+0$, $\Delta_3(z)\to 1$ as $z\to+\infty$. Therefore, the above of values $M^3_{\Lambda_1}$ function $\Delta_\nu(z)$ cannot vanish.

Let $\nu=3$ and $U>0$. Then the function $\Delta_\nu(z)$ are monotonically increasing function of z the outside of continuous spectrum of the operator $\widetilde{H}^1_{2\Lambda_1}$, i.e., in $(-\infty, m^3_{\Lambda_1})$ and in $(M^3_{\Lambda_1}, +\infty)$. For $z<m^3_{\Lambda_1}$ the function $\Delta_3(z)$ increasing from 1 to $1+\frac{UM}{4B}>1$, $\Delta_3(z)\to 1$ as $z\to-\infty$, $\Delta_3(z)\to 1+\frac{UM}{4B}$ as $z\to m^3_{\Lambda_1}-0$. Therefore, the below of values $m^3_{\Lambda_1}$ function $\Delta_\nu(z)$ cannot vanish. For $z>M^3_{\Lambda_1}$ the function $\Delta_3(z)$ increasing from $-\infty$ to $1-\frac{Um}{4B}$, $\Delta_3(z)\to 1-\frac{Um}{4B}$ as $z\to M^3_{\Lambda_1}+0$, $\Delta_2(z)\to 1$ as $z\to+\infty$. Therefore, the above of values $M^3_{\Lambda_1}$ the function $\Delta_\nu(z)$ vanishes at a single point $z_1^{''}$, if $1-\frac{Um}{4B}<0$, i.e., $U>\frac{4B}{m}$. □

We consider the Watson integral [16]

$$W = \frac{1}{\pi^3}\int\limits_0^\pi\int\limits_0^\pi\int\limits_0^\pi \frac{3\,dx\,dy\,dz}{3-\cos x-\cos y-\cos z} \approx 1,516.$$

Because the measure $d\lambda$ is normalized,

$$J=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dxdydz}{3-\cos x-\cos y-\cos z}=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dxdydz}{3+\cos x+\cos y+\cos z}=\frac{W}{3}.$$

We now consider the case, $\nu=3$ and the total quasimomentum Λ_1 of the system have the form $\Lambda_1=\left(\Lambda_1^1,\Lambda_1^2,\Lambda_1^3\right)=\left(\Lambda_1^0,\Lambda_1^0,\Lambda_1^0\right)$.

Theorem 6. *Let* $\nu=3$ *and* $\Lambda_1=\left(\Lambda_1^0,\Lambda_1^0,\Lambda_1^0\right)$. *Then*

a). If $U<0$ *and* $U<-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, *then the operator* $\widetilde{H}_{2\Lambda_1}^1$ *has a unique bound state* φ *with the energy value* $\widetilde{\widetilde{z}}_1$, *that is below the continuous spectrum of operator* $\widetilde{H}_{2\Lambda_1}^1$, *i.e.,* $\widetilde{\widetilde{z}}_1<m_{\Lambda_1}^3$.

b). If $U<0$ *and* $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}\leq U<0$, *then the operator* $\widetilde{H}_{2\Lambda_1}^1$ *has no bound state with the energy value, that is below the continuous spectrum of operator* $\widetilde{H}_{2\Lambda_1}^1$.

c). If $U>0$ *and* $U>\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, *then the operator* $\widetilde{H}_{2\Lambda_1}^1$ *has a unique antibound state* φ *with the energy value* $z_1^{''}$, *that is above the continuous spectrum of operator* $\widetilde{H}_{2\Lambda_1}^1$, *i.e.,* $z_1^{''}>M_{\Lambda_1}^3$.

d). If $U>0$ *and* $0<U\leq\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, *then the operator* $\widetilde{H}_{2\Lambda_1}^1$ *has no antibound state with the energy value, that is above the continuous spectrum of operator* $\widetilde{H}_{2\Lambda_1}^1$.

P r o o f. Let $\nu=3$, $\Lambda_1=\left(\Lambda_1^1,\Lambda_1^2,\Lambda_1^3\right)=\left(\Lambda_1^0,\Lambda_1^0,\Lambda_1^0\right)$ and $U<0$. Then the continuous spectrum of operator $\widetilde{H}_{2\Lambda_1}^1$ consists of segment $\left[-2A-12B\cos\frac{\Lambda_1^0}{2},-2A+12B\cos\frac{\Lambda_1^0}{2}\right]$. In the three-dimensional case, at $U<0$ the function $\Delta_3(z)$ are monotonically decreasing function of z the outside of continuous spectrum of operator $\widetilde{H}_{2\Lambda_1}^1$, i.e., in $(-\infty,m_{\Lambda_1}^3)$ and in $(M_{\Lambda_1}^3,+\infty)$. For $z<m_{\Lambda_1}^3$ the function $\Delta_3(z)$ decreasing from 1 to $1+\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}$, $\Delta_3(z)\to 1$ as $z\to-\infty$, $\Delta_3(z)\to 1+\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}$ as $z\to m_{\Lambda_1}^3-0$. Therefore, the below of values $m_{\Lambda_1}^3$ the function $\Delta_\nu(z)$ has a single zero at the point $\widetilde{\widetilde{z}}_1$, if $1+\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}<0$, i.e., $U<-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$. For $z>M_{\Lambda_1}^3$ the function $\Delta_3(z)$ decreasing from $1-\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}>1$ to 1, $\Delta_3(z)\to 1-\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}$ as $z\to M_{\Lambda_1}^3+0$, $\Delta_3(z)\to 1$ as $z\to+\infty$. Therefore, the above of values $M_{\Lambda_1}^3$ the function $\Delta_3(z)$ cannot vanish.

Let $\nu=3$ and $U>0$. Then the function $\Delta_3(z)$ are monotonically increasing function of z the outside of continuous spectrum of the operator $\widetilde{H}_{2\Lambda_1}^1$, i.e., in $(-\infty,m_{\Lambda_1}^3)$ and in $(M_{\Lambda_1}^3,+\infty)$. For $z<m_{\Lambda_1}^3$ the function $\Delta_3(z)$ increasing from 1 to $1+\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}>1$, $\Delta_3(z)\to 1$ as $z\to-\infty$, $\Delta_3(z)\to 1+\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}$ as $z\to m_{\Lambda_1}^3-0$. Therefore, the below of values $m_{\Lambda_1}^3$ the function $\Delta_\nu(z)$ cannot vanish. For $z>M_{\Lambda_1}^3$ the function $\Delta_3(z)$ increasing from $-\infty$ to $1-\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}$, $\Delta_3(z)\to 1-\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}$ as $z\to M_{\Lambda_1}^3+0$, $\Delta_3(z)\to 1$ as $z\to+\infty$.

Therefore, the above of values $M^3_{\Lambda_1}$ the function $\Delta_\nu(z)$ has a single zero at the point $z_1^{''}$, if $1-\frac{UW}{12B\cos\frac{\Lambda_1^0}{2}}<0$, i.e., $U>\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$. □

Now we consider the operator

$$\left(\widetilde{H}_2^2\widetilde{f}\right)(\mu,\theta)=\left\{A+2B\sum_{i=1}^{\nu}\cos\mu_i\right\}\widetilde{f}(\mu,\theta)-U\int\limits_{T^\nu}\widetilde{f}(s,\mu+\theta-s)ds,$$

and investigate the spectrum of this operator.

Let the total quasimomentum $\Lambda_2=\mu+\theta$ of the system be fixed. Then the operator $\widetilde{H}_2^2$ takes the form

$$\left(\widetilde{H}^2_{2\Lambda_2}\widetilde{f}_{\Lambda_2}\right)(\mu)=\left\{A+2B\sum_{i=1}^{\nu}\cos\mu_i\right\}\widetilde{f}_{\Lambda_2}(\mu)-U\int\limits_{T^\nu}\widetilde{f}_{\Lambda_2}(s)ds,$$

where $\widetilde{f}_{\Lambda_2}(s)=\widetilde{f}(s,\Lambda_2-s)$.

We set

$$\widetilde{\Delta}_\nu(z)=1-U\int\limits_{T^\nu}\frac{ds_1ds_2...ds_\nu}{A+2B\sum_{i=1}^{\nu}\cos s_i-z}.$$

Lemma 3. *A number $z=z_0\notin\sigma_{cont}(\widetilde{H}^2_{2\Lambda_2})$, is an eigenvalue of operator $\widetilde{H}^2_{2\Lambda_2}$ if and only if it is a zero of the function $\widetilde{\Delta}_\nu(z)$, i.e., $\widetilde{\Delta}_\nu(z_0)=0$.*

It is known that the continuous spectrum of operator $\widetilde{H}^2_{2\Lambda_2}$ fills the entire interval $\left[m^\nu_{\Lambda_2},M^\nu_{\Lambda_2}\right]=[A-2B\nu,A+2B\nu]$.

We consider one-dimensional case.

Theorem 7. *At values $\nu=1$ and $U<0$ ($U>0$), and for all values of parameters of the Hamiltonian, the operator $\widetilde{H}^2_{2\Lambda_2}$ has a unique two-electron bound state (antibound state) φ whit the energy value $z_2=A+\sqrt{U^2+4B^2}$ $\left(\widetilde{z}_2=A-\sqrt{U^2+4B^2}\right)$, that is above (below) the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$, i.e., $z_2>M^1_{\Lambda_2}$ $\left(\widetilde{z}_2<m^1_{\Lambda_2}\right)$.*

In two-dimensional case, we have the analogously results.

We consider three-dimensional case.

Theorem 8. *a). If $U<0$ and $U<-\frac{6B}{W}$, then the operator $\widetilde{H}^2_{2\Lambda_2}$ has a unique bound state φ with the energy value $\widetilde{\widetilde{z}}_2$, that is above the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$, i.e., $\widetilde{\widetilde{z}}_2>M^3_{\Lambda_2}$.*

b). If $U<0$ and $-\frac{6B}{W}\leq U<0$, then the operator $\widetilde{H}^2_{2\Lambda_2}$ has no bound state with the energy value, that is above the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$.

c). If $U>0$ and $U>\frac{6B}{W}$, then the operator $\widetilde{H}^2_{2\Lambda_2}$ has a unique antibound state φ with the energy value $z_2^{''}$, that is below the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$, i.e., $z_2^{''}<m^3_{\Lambda_2}$.

d). If $U>0$ and $0<U\leq\frac{6B}{W}$, then the operator $\widetilde{H}^2_{2\Lambda_2}$ has no antibound state with the energy value, that is below the continuous spectrum of the operator $\widetilde{H}^2_{2\Lambda_2}$.

Now we consider the operator

$$\left(\widetilde{H}_2^3\widetilde{f}\right)(\theta,\eta)=\left\{2A+2B\sum_{i=1}^{\nu}\left[\cos\theta_i+\cos\eta_i\right]\right\}\widetilde{f}(\theta,\eta)+U\int_{T^\nu}\widetilde{f}(s,\theta+\eta-s)ds+U\int_{T^\nu}\widetilde{f}(s,\gamma+\eta-s)ds,$$

and investigate the spectrum of this operator.

Let the total quasimomentum $\Lambda_3=\theta+\eta$ and $\Lambda_4=\gamma+\eta$. Then the operator $\widetilde{H}_2^3$ takes the form

$$\left(\widetilde{H}_{2\Lambda_3}^3\widetilde{f}_{\Lambda_3}\right)(\theta)=\left\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2}\cos\left(\frac{\Lambda_3^i}{2}-\theta_i\right)\right\}\widetilde{f}_{\Lambda_3}(\theta)+U\int_{T^\nu}\widetilde{f}_{\Lambda_3}(s)ds+U\int_{T^\nu}\widetilde{f}_{\Lambda_4}(s)ds,$$

where $\widetilde{f}_{\Lambda_3}(s)=\widetilde{f}(s,\Lambda_3-s)$, and $\widetilde{f}_{\Lambda_4}(s)=\widetilde{f}(s,\Lambda_4-s)$.

It is known that the regions of change of parameters Λ_3 and Λ_4 identical, therefore, the action of operators $\widetilde{H}_{2\Lambda_3}^3$ it is possible write in the next form

$$\left(\widetilde{H}_{2\Lambda_3}^3\widetilde{f}_{\Lambda_3}\right)(\theta)=\left\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2}\cos\left(\frac{\Lambda_3^i}{2}-\theta_i\right)\right\}\widetilde{f}_{\Lambda_3}(\theta)+2U\int_{T^\nu}\widetilde{f}_{\Lambda_3}(s)ds.$$

We set

$$\widetilde{\widetilde{\Delta}}_\nu(z)=1+2U\int_{T^\nu}\frac{ds_1ds_2\dots ds_\nu}{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2}\cos\left(\frac{\Lambda_3^i}{2}-s_i\right)-z}.$$

It is known that the continuous spectrum of operator $\widetilde{H}_{2\Lambda_3}^3$ is independent of the parameter U and consists of the segments $\left[2A-4B\sum\limits_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2},2A+4B\sum\limits_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2}\right]$.

Lemma 4. *A number $z=z_0\notin\sigma_{cont}(\widetilde{H}_{2\Lambda_3}^3)$ is an eigenvalue of operator $\widetilde{H}_{2\Lambda_3}^3$ if and only if it is a zero of the function $\widetilde{\widetilde{\Delta}}_\nu(z)$, i.e., $\widetilde{\widetilde{\Delta}}_\nu(z_0)=0$.*

We consider the one-dimensional case.

Theorem 9. *If $\nu=1$ and $U<0$ ($U>0$), and for all values of parameters of the Hamiltonian, the operator $\widetilde{H}_{2\Lambda_3}^3$ has a unique two-electron bound state (antibound state) φ with the energy value $z_3=2A-\sqrt{4U^2+16B^2\cos^2\frac{\Lambda_3}{2}}$ $\left(\widetilde{z}_3=2A+\sqrt{4U^2+16B^2\cos^2\frac{\Lambda_3}{2}}\right)$, that is below (above) the continuous spectrum of the operator $\widetilde{H}_{2\Lambda_3}^3$, i.e., $z_3<m_{\Lambda_3}^1$ $\left(\widetilde{z}_3>M_{\Lambda_3}^1\right)$.*

In the two-dimensional case, we have analogously results.

Now we consider three-dimensional case. We denote

$$M=\int_{T^3}\frac{ds_1ds_2ds_3}{\sum_{i=1}^3\cos\frac{\Lambda_3^i}{2}\left(1-\cos\left(\frac{\Lambda_3^i}{2}-s_i\right)\right)}\quad\text{and}\quad m=\int_{T^3}\frac{ds_1ds_2ds_3}{\sum_{i=1}^3\cos\frac{\Lambda_3^i}{2}\left(1+\cos\left(\frac{\Lambda_3^i}{2}-s_i\right)\right)}.$$

Let $\nu=3$.

Theorem 10. *a). If $U<0$ and $U<-\frac{2B}{m}$, then the operator $\widetilde{H}^3_{2\Lambda_3}$ has a unique bound state φ with the energy value $\widetilde{\widetilde{z}}_3$, that is below the continuous spectrum of the operator $\widetilde{H}^3_{2\Lambda_3}$, i.e., $\widetilde{\widetilde{z}}_3<m^3_{\Lambda_3}$.*

b). If $U<0$ and $-\frac{2B}{m}\le U<0$, then the operator $\widetilde{H}^3_{2\Lambda_3}$ has no bound state with the energy value, that is below the continuous spectrum of the operator $\widetilde{H}^3_{2\Lambda_3}$.

c). If $U>0$ and $U>\frac{2B}{M}$, then the operator $\widetilde{H}^3_{2\Lambda_3}$ has a unique antibound state φ with the energy value $z_3{}''$, that is above the continuous spectrum of the operator $\widetilde{H}^3_{2\Lambda_3}$, i.e., $z_3{}''>M^3_{\Lambda_3}$.

d). If $U>0$ and $0<U\le\frac{2B}{M}$, then the operator $\widetilde{H}^3_{2\Lambda_3}$ has no antibound state with the energy value, that is above the continuous spectrum of the operator $\widetilde{H}^3_{2\Lambda_3}$.

Now we consider the case, when $\nu=3$ and the total quasimomentum Λ_3 of the system has the form $\Lambda_3=\left(\Lambda_3^1,\Lambda_3^2,\Lambda_3^3\right)=\left(\Lambda_3^0,\Lambda_3^0,\Lambda_3^0\right)$.

Theorem 11. *Let $\nu=3$ and $\Lambda_3=\left(\Lambda_3^0,\Lambda_3^0,\Lambda_3^0\right)$. Then*

a). If $U<0$ and $U<-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, then the operator $\widetilde{H}^3_{2\Lambda_3}$ has a unique bound state φ with the energy value $\widetilde{\widetilde{z}}_3$, that is below the continuous spectrum of the operator $\widetilde{H}^3_{2\Lambda_3}$, i.e., $\widetilde{\widetilde{z}}_3<m^3_{\Lambda_3}$.

b). If $U<0$ and $-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}\le U<0$, then the operator $\widetilde{H}^3_{2\Lambda_3}$ has no bound state with the energy value, that is below the continuous spectrum of the operator $\widetilde{H}^3_{2\Lambda_3}$.

c). If $U>0$ and $U>\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, then the operator $\widetilde{H}^3_{2\Lambda_3}$ has a unique antibound state φ with the energy value $z_3{}''$, that is above the continuous spectrum of the operator $\widetilde{H}^3_{2\Lambda_3}$, i.e., $z_3{}''>M^3_{\Lambda_3}$.

d). If $U>0$ and $0<U\le\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, then the operator $\widetilde{H}^3_{2\Lambda_3}$ has no antibound state with the energy value, that is above the continuous spectrum of the operator $\widetilde{H}^3_{2\Lambda_3}$.

The spectrum of the operator $A\bigotimes I+I\bigotimes B$, where A and B are densely defined bounded linear operators, was studied in [17–19]. Explicit formulas were given there that express the essential spectrum $\sigma_{ess}(A\bigotimes I+I\bigotimes B)$ and discrete spectrum $\sigma_{disc}(A\bigotimes I+I\bigotimes B)$ of operator $A\bigotimes I+I\bigotimes B$ in terms of the spectrum $\sigma(A)$ and the discrete spectrum $\sigma_{disc}(A)$ of A and in terms of the spectrum $\sigma(B)$ and discrete spectrum $\sigma_{disc}(B)$ of B:

$$\sigma_{disc}\left(A\bigotimes I+I\bigotimes B\right)=$$
$$=\Big\{\sigma(A)\backslash\sigma_{ess}(A)+\sigma(B)\backslash\sigma_{ess}(B)\Big\}\backslash\Big\{(\sigma_{ess}(A)+\sigma(B))\cup(\sigma(A)+\sigma_{ess}(B))\Big\},$$
$$\sigma_{ess}\left(A\bigotimes I+I\bigotimes B\right)=\Big(\sigma_{ess}(A)+\sigma(B)\Big)\cup\Big(\sigma(A)+\sigma_{ess}(B)\Big).$$

It is clear that $\sigma\left(A\bigotimes I+I\bigotimes B\right)=\Big\{\lambda+\mu:\lambda\in\sigma(A),\mu\in\sigma(B)\Big\}$.

We now using the obtaining results and representation (5), we can describe the structure of essential spectrum and discrete spectrum of the operator of four five-electron quartet state of the system ${}^4\widetilde{H}^q_{3/2}$.

2 Essential spectra and discrete spectrum of the operator of fourth five-electron quartet state of the system ${}^4\widetilde{H}^q_{3/2}$

Theorem 12. *a). Let $\nu = 1$ and $U < 0$. Then the essential spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ of the system in a fourth five-electron quartet state is exactly the union of seven segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = [a+c+e, b+d+f] \cup [a+c+z_3, b+d+z_3] \cup [a+e+z_2, b+f+z_2] \cup$$
$$\cup [a+z_2+z_3, b+z_2++z_3] \cup [c+z_1+z_3, d+z_1+z_3] \cup$$
$$\cup [c+e+z_1, d+f+z_1] \cup [e+z_1+z_2, f+z_1+z_2].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ consists of no more then one point: or $\sigma_{disc}({}^4\widetilde{H}^q_{3/2}) = \emptyset$, or $\sigma_{disc}({}^4\widetilde{H}^q_{3/2}) = \{z_1+z_2+z_3\}$, here and hereafter $a = -2A - 4B\cos\frac{\Lambda_1}{2}$, $b = -2A + 4B\cos\frac{\Lambda_1}{2}$, $c = A - 2B$, $d = A + 2B$, $e = 2A - 4B\cos\frac{\Lambda_3}{2}$, $f = 2A + 4B\cos\frac{\Lambda_3}{2}$, $z_1 = -2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$, $z_2 = A + \sqrt{U^2 + 4B^2}$, and $z_3 = 2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_3}{2}}$.

b). Let $\nu = 1$ and $U > 0$. Then the essential spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ of the system in a fourth five-electron quartet state is exactly the union of seven segments,

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = [a+c+e, b+d+f] \cup [a+c+\widetilde{z}_3, b+d+\widetilde{z}_3] \cup [a+e+\widetilde{z}_2, b+f+\widetilde{z}_2] \cup$$
$$\cup [a+\widetilde{z}_2+\widetilde{z}_3, b+\widetilde{z}_2++\widetilde{z}_3] \cup [c+\widetilde{z}_1+\widetilde{z}_3, d+\widetilde{z}_1+\widetilde{z}_3] \cup$$
$$\cup [c+e+\widetilde{z}_1, d+f+\widetilde{z}_1] \cup [e+\widetilde{z}_1+\widetilde{z}_2, f+\widetilde{z}_1+\widetilde{z}_2].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ consists of no more then one point: or $\sigma_{disc}({}^4\widetilde{H}^q_{3/2}) = \emptyset$, or $\sigma_{disc}({}^4\widetilde{H}^q_{3/2}) = \{\widetilde{z}_1+\widetilde{z}_2+\widetilde{z}_3\}$. Here $\widetilde{z}_1 = -2A + \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$, $\widetilde{z}_2 = A - \sqrt{U^2+4B^2}$, and $\widetilde{z}_3 = 2A + \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_3}{2}}$.

Proof. It follows from representation (5) that

$$\sigma\left({}^4\widetilde{H}^q_{3/2}\right) = \left\{\lambda+\mu+\theta : \lambda \in \sigma(\widetilde{H}^1_{2\Lambda_1}), \mu \in \sigma(\widetilde{H}^2_{2\Lambda_2}), \theta \in \sigma(\widetilde{H}^3_{2\Lambda_3})\right\},$$

and one-dimensional case, if $U < 0$, then the continuous spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$ consists of segment $\left[m^1_{\Lambda_1}, M^1_{\Lambda_1}\right] = \left[-2A - 4B\cos\frac{\Lambda_1}{2}, -2A + 4B\cos\frac{\Lambda_1}{2}\right]$, and the discrete spectrum of operator $\widetilde{H}^1_{2\Lambda_1}$ consists of single point $z_1 = -2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$. The continuous spectrum of operator $\widetilde{H}^2_{2\Lambda_2}$ consists of segment $\left[m^1_{\Lambda_2}, M^1_{\Lambda_2}\right] = [A-2B, A+2B]$, and the discrete spectrum of operator $\widetilde{H}^2_{2\Lambda_2}$ consists of single point $z_2 = A + \sqrt{U^2+4B^2}$. The continuous spectrum of operator $\widetilde{H}^3_{2\Lambda_3}$ is consists of segment $\left[2A - 4B\cos\frac{\Lambda_3}{2}, 2A + 4B\cos\frac{\Lambda_3}{2}\right]$, and the discrete spectrum of operator $\widetilde{H}^3_{2\Lambda_3}$ consists of single point $z_3 = 2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_3}{2}}$. Therefore, the essential spectrum of the system of fourth five-electron quartet state operator ${}^4\widetilde{H}^q_{3/2}$ is the union of seven segments, and the fourth five-electron quartet state operator ${}^4\widetilde{H}^q_{3/2}$ has eigenvalue $\{z_1+z_2+z_3\}$. If $\{z_1+z_2+z_3\}\in\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)$, then the discrete spectrum of operator ${}^4\widetilde{H}^q_{3/2}$ empty set: $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\emptyset$, if $\{z_1+z_2+z_3\}\notin\sigma_{ess}({}^4\widetilde{H}^q_{3/2})$, then the discrete spectrum of operator ${}^4\widetilde{H}^q_{3/2}$ consists of unique eigenvalue $\{z_1+z_2+z_3\}$, i.e., $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\{z_1+z_2+z_3\}$. From here we find the statement a) of theorem 12. The statement b) of theorem 12 is proved a similarly. □

In the two-dimensional case the similar results occur.

We now consider the three-dimensional case.

In the three-dimensional case, the structure of essential spectra and discrete spectrum of operator ${}^4\widetilde{H}^q_{3/2}$ is described by the following theorems:

Theorem 13. *Let $\nu=3$ and $U<0$. Then the following statement is holds.*

a). Let $U<-\frac{6B}{W}$, $M>\frac{2}{3}W$ and $m>\frac{M}{2}$, ($m<\frac{M}{2}$,) or $U<-\frac{4B}{M}$, $M<\frac{2}{3}W$ and $m>\frac{1}{3}W$, or $U<-\frac{4B}{M}$, $m>\frac{1}{2}M$, and $m<\frac{1}{3}W$, or $U<-\frac{2B}{m}$, $m<\frac{1}{3}W$, and $M>\frac{2}{3}W$ or $U<-\frac{2B}{m}$, $M<\frac{2}{3}W$, and $m<\frac{1}{2}M$, then the essential spectrum of the system fourth five-electron quartet state operator ${}^4\widetilde{H}^q_{3/2}$ is the union of seven segments,

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+\widetilde{\widetilde{z}}_3,b_1+d_1+\widetilde{\widetilde{z}}_3\right]\cup$$
$$\cup\left[a_1+e_1+\widetilde{\widetilde{z}}_2,b_1+f_1+\widetilde{\widetilde{z}}_2\right]\cup\left[a_1+\widetilde{\widetilde{z}}_2+\widetilde{\widetilde{z}}_3,b_1+\widetilde{\widetilde{z}}_2+\widetilde{\widetilde{z}}_3\right]\cup$$
$$\cup\left[c_1+\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_3,d_1+\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_3\right]\cup\left[c_1+e_1+\widetilde{\widetilde{z}}_1,d_1+f_1+\widetilde{\widetilde{z}}_1\right]\cup\left[e_1+\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_2,f_1+\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_2\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ consists of no more one point: or $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\emptyset$, or $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\left\{\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_2+\widetilde{\widetilde{z}}_3\right\}$, where

$$a_1=-2A-4B\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2},\quad b_1=-2A+4B\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2},\quad c_1=A-6B,$$
$$d_1=A+6B,\quad e_1=2A-4B\sum_{i=1}^{3}\cos\frac{\Lambda_3^i}{2},\quad f_1=2A+4B\sum_{i=1}^{3}\cos\frac{\Lambda_3^i}{2},$$

$\widetilde{\widetilde{z}}_1$, $\widetilde{\widetilde{z}}_2$, and $\widetilde{\widetilde{z}}_3$ are the eigenvalues of the operators $\widetilde{H}^1_{2\Lambda_1}$, $\widetilde{H}^2_{2\Lambda_2}$, and $\widetilde{H}^3_{2\Lambda_3}$, correspondingly.

b). Let $-\frac{6B}{W}\le U<-\frac{4B}{M}$, $m>\frac{1}{2}M$, and $M>\frac{2}{3}W$, or $-\frac{6B}{W}\le U<-\frac{2B}{m}$, $m<\frac{1}{2}M$, and $m>\frac{1}{3}W$, or $-\frac{4B}{M}\le U<-\frac{6B}{M}$, $m>\frac{1}{3}W$, and $M<\frac{2}{3}W$, or $-\frac{4B}{M}\le U<-\frac{2B}{m}$, $m<\frac{1}{3}W$, and $m>\frac{1}{2}M$, or $-\frac{2B}{m}\le U<-\frac{6B}{M}$, $M>\frac{2}{3}W$, and $m<\frac{1}{3}W$, or $-\frac{2B}{m}\le U<-\frac{4B}{M}$, $M<\frac{2}{3}W$, and $m<\frac{1}{2}M$. Then the essential spectrum of the system fourth five-electron quartet state operator ${}^4\widetilde{H}^q_{3/2}$ is the union of four segments,

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+\widetilde{\widetilde{z}}_3,b_1+d_1+\widetilde{\widetilde{z}}_3\right]\cup$$
$$\cup\left[c_1+\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_3,d_1+\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_3\right]\cup\left[c_1+e_1+\widetilde{\widetilde{z}}_1,d_1+f_1+\widetilde{\widetilde{z}}_1\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[a_1+c_1+e_1, b_1+d_1+f_1\right] \cup \left[a_1+c_1+\widetilde{\widetilde{z}}_3, b_1+d_1+\widetilde{\widetilde{z}}_3\right] \cup$$
$$\cup \left[a_1+\widetilde{\widetilde{z}}_2+\widetilde{\widetilde{z}}_3, b_1+\widetilde{\widetilde{z}}_2+\widetilde{\widetilde{z}}_3\right] \cup \left[a_1+e_1+\widetilde{\widetilde{z}}_2, b_1+f_1+\widetilde{\widetilde{z}}_2\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[a_1+c_1+e_1, b_1+d_1+f_1\right] \cup \left[a_1+e_1+\widetilde{\widetilde{z}}_2, b_1+f_1+\widetilde{\widetilde{z}}_2\right] \cup$$
$$\cup \left[e_1+\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_2, f_1+\widetilde{\widetilde{z}}_1+\widetilde{\widetilde{z}}_2\right] \cup \left[c_1+e_1+\widetilde{\widetilde{z}}_1, d_1+f_1+\widetilde{\widetilde{z}}_1\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \emptyset$.

c). Let $-\frac{4B}{M} \le U < -\frac{2B}{m}$, $M > \frac{2}{3}W$, *and* $m > \frac{1}{2}M$, *or* $-\frac{2B}{m} \le U < -\frac{4B}{M}$, $m > \frac{1}{3}W$, *and* $m < \frac{1}{2}M$, *or* $-\frac{6B}{W} \le U < -\frac{2B}{m}$, $M < \frac{2}{3}W$, *and* $m > \frac{1}{3}W$, *or* $-\frac{2B}{m} \le U < -\frac{6B}{W}$, $m > \frac{1}{2}M$, *and* $m < \frac{1}{3}W$, *or* $-\frac{6B}{W} \le U < -\frac{4B}{M}$, $m < \frac{1}{3}W$, *and* $M > \frac{2}{3}W$, *or* $-\frac{4B}{M} \le U < -\frac{6B}{W}$, $m < \frac{1}{2}M$, *and* $M < \frac{2}{3}W$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of two segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[a_1+c_1+e_1, b_1+d_1+f_1\right] \cup \left[a_1+c_1+\widetilde{\widetilde{z}}_3, b_1+d_1+\widetilde{\widetilde{z}}_3\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[a_1+c_1+e_1, b_1+d_1+f_1\right] \cup \left[c_1+e_1+\widetilde{\widetilde{z}}_1, d_1+f_1+\widetilde{\widetilde{z}}_1\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[a_1+c_1+e_1, b_1+d_1+f_1\right] \cup \left[a_1+e_1+\widetilde{\widetilde{z}}_2, b_1+f_1+\widetilde{\widetilde{z}}_2\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \emptyset$.

d). Let $-\frac{2B}{m} \le U < 0$, $M > \frac{2}{3}W$ $(M < \frac{2}{3}W)$ *and* $m > \frac{1}{2}M$, *or* $-\frac{4B}{M} \le U < 0$, $m > \frac{1}{3}W$ $(m < \frac{1}{3}W)$ *and* $m < \frac{1}{2}M$, *or* $-\frac{6B}{W} \le U < 0$, $m > \frac{1}{2}M$ $(m < \frac{1}{2}M)$ *and* $m < \frac{1}{3}W$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is consists of a single segment:* $\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[a_1+c_1+e_1, b_1+d_1+f_1\right]$. *The discrete spectrum of the operator* ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \emptyset$.

Theorem 14. *Let* $U > 0$. *Then the following statement is holds.*

a). Let $U > \frac{6B}{W}$, $m > \frac{2}{3}W$ *and* $m < 2M$, *or* $U > \frac{6B}{W}$, $m > 2M$, *and* $M > \frac{1}{3}W$, *or* $U > \frac{4B}{m}$, $m < \frac{2}{3}W$ *and* $M > \frac{1}{3}W$, *or* $U > \frac{4B}{m}$, $M < \frac{1}{3}W$ *and* $m < 2M$, *or* $U > \frac{2B}{M}$, $M < \frac{1}{3}W$ *and* $m > \frac{2}{3}W$, *or* $U > \frac{2B}{M}$, $m < \frac{2}{3}W$, *and* $m > \frac{1}{2}M$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of seven segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[a_1+c_1+e_1, b_1+d_1+f_1\right] \cup \left[a_1+c_1+z_3^{''}, b_1+d_1+z_3^{''}\right] \cup$$
$$\cup \left[a_1+e_1+z_2^{''}, b_1+f_1+z_2^{''}\right] \cup \left[a_1+z_2^{''}+z_3^{''}, b_1+z_2^{''}+z_3^{''}\right] \cup \left[c_1+z_1^{''}+z_3^{''}, d_1+z_1^{''}+z_3^{''}\right] \cup$$
$$\cup \left[c_1+e_1+z_1^{''}, d_1+f_1+z_1^{''}\right] \cup \left[e_1+z_1^{''}+z_2^{''}, f_1+z_1^{''}+z_2^{''}\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *consists of no more one point: or* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\emptyset$, *or* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\left\{z_1^{''}+z_2^{''}+z_3^{''}\right\}$, *where* $z_1^{''}$, $z_2^{''}$, *and* $z_3^{''}$ *are the eigenvalues of the operators* $\widetilde{H}^1_{2\Lambda_1}$, $\widetilde{H}^2_{2\Lambda_2}$, *and* $\widetilde{H}^3_{2\Lambda_3}$, *correspondingly.*

b). Let $\frac{6B}{W}<U\leq\frac{2B}{M}$, $m>\frac{2}{3}W$, *and* $M<\frac{1}{3}W$, *or* $\frac{2B}{M}<U\leq\frac{6B}{W}$, $m>2M$, *and* $M>\frac{1}{3}W$, *or* $\frac{4B}{m}<U\leq\frac{2B}{M}$, $m>2M$, *and* $m<\frac{2}{3}W$, *or* $\frac{2B}{M}<U\leq\frac{4B}{m}$, $m<2M$, *and* $M<\frac{1}{3}W$, *or* $\frac{4B}{m}<U\leq\frac{6B}{W}$, $m>\frac{2}{3}W$, *and* $m<2M$, *or* $\frac{6B}{W}<U\leq\frac{4B}{m}$, $m<\frac{2}{3}W$, *and* $M>\frac{1}{3}W$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of four segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+e_1+z_2^{''},b_1+f_1+z_2^{''}\right]\cup$$
$$\cup\left[c_1+e_1+z_1^{''},d_1+f_1+z_1^{''}\right]\cup\left[e_1+z_1^{''}+z_2^{''},f_1+z_1^{''}+z_2^{''}\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3^{''},b_1+d_1+z_3^{''}\right]\cup$$
$$\cup\left[c_1+e_1+z_1^{''},d_1+f_1+z_1^{''}\right]\cup\left[c_1+z_1^{''}+z_3^{''},d_1+z_1^{''}+z_3^{''}\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+e_1+z_2^{''},b_1+f_1+z_2^{''}\right]\cup$$
$$\cup\left[a_1+c_1+z_3^{''},b_1+d_1+z_3^{''}\right]\cup\left[a_1+z_2^{''}+z_3^{''},b_1+z_2^{''}+z_3^{''}\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\emptyset$.

c). Let $\frac{4B}{m}<U\leq\frac{6B}{W}$, $m>\frac{2}{3}W$, *and* $M<\frac{1}{3}W$, *or* $\frac{4B}{m}<U\leq\frac{2B}{M}$, $m>2M$, *and* $m>\frac{2}{3}W$, *or* $\frac{6B}{W}<U\leq\frac{4B}{m}$, $m<\frac{2}{3}W$, *and* $m>2M$, *or* $\frac{6B}{W}<U\leq\frac{2B}{M}$, $M<\frac{1}{3}W$, *and* $m<2M$, *or* $\frac{2B}{M}<U\leq\frac{4B}{m}$, $m<2M$, *and* $m>\frac{2}{3}W$, *or* $\frac{2B}{M}<U\leq\frac{6B}{W}$, $M>\frac{1}{3}W$, *and* $m<\frac{2}{3}W$, *then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of two segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[c_1+e_1+z_1^{''},d_1+f_1+z_1^{''}\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+e_1+z_2^{''},b_1+f_1+z_2^{''}\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3^{''},b_1+d_1+z_3^{''}\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\emptyset$.

d). Let $0<U\leq\frac{4B}{m}$, $m>\frac{2}{3}W$, *and* $M<\frac{1}{3}W$, *or* $m>2M$, *and* $M>\frac{1}{3}W$, *or* $0<U\leq\frac{6B}{W}$, $m<\frac{2}{3}W$, *and* $m>2M$, *or* $M<\frac{1}{3}W$, *and* $m<2M$, *or* $0<U\leq\frac{2B}{M}$, $m<2M$, *and* $m>\frac{2}{3}W$, *or* $M>\frac{1}{3}W$, *and* $m<\frac{2}{3}W$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *consists of a single segment:*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \emptyset$.

Let $\nu = 3$ and $\Lambda_1 = \left(\Lambda_1^0, \Lambda_1^0, \Lambda_1^0\right)$, and $\Lambda_3 = \left(\Lambda_3^0, \Lambda_3^0, \Lambda_3^0\right)$.

Theorem 15. *Let* $U < 0$. *Then the following statement is holds.*

a). Let $U < 0$, $U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U < 0$, $U < -\frac{6B}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$ *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of seven segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[\widetilde{a}_1 + \widetilde{c}_1 + \widetilde{e}_1, \widetilde{b}_1 + \widetilde{d}_1 + \widetilde{f}_1\right] \cup \left[\widetilde{a}_1 + \widetilde{c}_1 + \widetilde{\widetilde{z}}_3, \widetilde{b}_1 + \widetilde{d}_1 + \widetilde{\widetilde{z}}_3\right] \cup$$
$$\cup \left[\widetilde{a}_1 + \widetilde{e}_1 + \widetilde{\widetilde{z}}_2, \widetilde{b}_1 + \widetilde{f}_1 + \widetilde{\widetilde{z}}_2\right] \cup \left[\widetilde{a}_1 + \widetilde{\widetilde{z}}_2 + \widetilde{\widetilde{z}}_3, \widetilde{b}_1 + \widetilde{\widetilde{z}}_2 + \widetilde{\widetilde{z}}_3\right] \cup$$
$$\cup \left[\widetilde{c}_1 + \widetilde{\widetilde{z}}_1 + \widetilde{\widetilde{z}}_3, \widetilde{d}_1 + \widetilde{\widetilde{z}}_1 + \widetilde{\widetilde{z}}_3\right] \cup \left[\widetilde{c}_1 + \widetilde{e}_1 + \widetilde{\widetilde{z}}_1, \widetilde{d}_1 + \widetilde{f}_1 + \widetilde{\widetilde{z}}_1\right] \cup \left[\widetilde{e}_1 + \widetilde{\widetilde{z}}_1 + \widetilde{\widetilde{z}}_2, \widetilde{f}_1 + \widetilde{\widetilde{z}}_1 + \widetilde{\widetilde{z}}_2\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *consists of no more one point: or* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \emptyset$, *or* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \left\{\widetilde{\widetilde{z}}_1 + \widetilde{\widetilde{z}}_2 + \widetilde{\widetilde{z}}_3\right\}$. *Here and hereafter*

$$\widetilde{a}_1 = -2A - 12B\cos\frac{\Lambda_1^0}{2}, \quad \widetilde{b}_1 = -2A + 12B\cos\frac{\Lambda_1^0}{2}, \quad \widetilde{c}_1 = A - 6B,$$
$$\widetilde{d}_1 = A + 6B, \quad \widetilde{e}_1 = 2A - 12B\cos\frac{\Lambda_3^0}{2}, \quad \widetilde{f}_1 = 2A + 12B\cos\frac{\Lambda_3^0}{2},$$

$\widetilde{\widetilde{z}}_1$, $\widetilde{\widetilde{z}}_2$, *and* $\widetilde{\widetilde{z}}_3$ *are the eigenvalues of the operators* $\widetilde{H}^1_{2\Lambda_1}$, $\widetilde{H}^2_{2\Lambda_2}$, *and* $\widetilde{H}^3_{2\Lambda_3}$, *correspondingly.*

b). Let $U < 0$, $-\frac{6B}{W} \leq U < -\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U < 0$, $-\frac{6B}{W} \leq U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U < 0$, $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < -\frac{6B}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of four segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[\widetilde{a}_1 + \widetilde{c}_1 + \widetilde{e}_1, \widetilde{b}_1 + \widetilde{d}_1 + \widetilde{f}_1\right] \cup \left[\widetilde{a}_1 + \widetilde{c}_1 + \widetilde{\widetilde{z}}_3, \widetilde{b}_1 + \widetilde{d}_1 + \widetilde{\widetilde{z}}_3\right] \cup$$
$$\cup \left[\widetilde{c}_1 + \widetilde{\widetilde{z}}_1 + \widetilde{\widetilde{z}}_3, \widetilde{d}_1 + \widetilde{\widetilde{z}}_1 + \widetilde{\widetilde{z}}_3\right] \cup \left[\widetilde{c}_1 + \widetilde{e}_1 + \widetilde{\widetilde{z}}_1, \widetilde{d}_1 + \widetilde{f}_1 + \widetilde{\widetilde{z}}_1\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[\widetilde{a}_1 + \widetilde{c}_1 + \widetilde{e}_1, \widetilde{b}_1 + \widetilde{d}_1 + \widetilde{f}_1\right] \cup \left[\widetilde{a}_1 + \widetilde{c}_1 + \widetilde{\widetilde{z}}_3, \widetilde{b}_1 + \widetilde{d}_1 + \widetilde{\widetilde{z}}_3\right] \cup$$
$$\cup \left[\widetilde{a}_1 + \widetilde{\widetilde{z}}_2 + \widetilde{\widetilde{z}}_3, \widetilde{b}_1 + \widetilde{\widetilde{z}}_2 + \widetilde{\widetilde{z}}_3\right] \cup \left[\widetilde{a}_1 + \widetilde{e}_1 + \widetilde{\widetilde{z}}_2, \widetilde{b}_1 + \widetilde{f}_1 + \widetilde{\widetilde{z}}_2\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}({}^4\widetilde{H}^q_{3/2}) = \emptyset$.

c). Let $U < 0$, $-\frac{6B}{W} \leq U < -\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, *or* $U < 0$, $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < -\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, *or* $U < 0$,

$-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} \leq U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of two segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[\widetilde{a}_1+\widetilde{c}_1+\widetilde{e}_1, \widetilde{b}_1+\widetilde{d}_1+\widetilde{f}_1\right] \cup \left[\widetilde{a}_1+\widetilde{c}_1+\widetilde{\widetilde{z}}_3, \widetilde{b}_1+\widetilde{d}_1+\widetilde{\widetilde{z}}_3\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[\widetilde{a}_1+\widetilde{c}_1+\widetilde{e}_1, \widetilde{b}_1+\widetilde{d}_1+\widetilde{f}_1\right] \cup \left[\widetilde{c}_1+\widetilde{e}_1+\widetilde{\widetilde{z}}_1, \widetilde{d}_1+\widetilde{f}_1+\widetilde{\widetilde{z}}_1\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \emptyset$.

d). Let $U < 0$, $-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} \leq U < 0$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, *or* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, *or* $U < 0$, $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < 0$, *and* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is consists of a single segment:* $\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[a_1+c_1+e_1, b_1+d_1+f_1\right]$. *The discrete spectrum of the operator* ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \emptyset$.

e). Let $U > 0$, $U > \frac{6B}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, $U > \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of seven segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[\widetilde{a}_1+\widetilde{c}_1+\widetilde{e}_1, \widetilde{b}_1+\widetilde{d}_1+\widetilde{f}_1\right] \cup \left[\widetilde{a}_1+\widetilde{c}_1+z_3^{''}, \widetilde{b}_1+\widetilde{d}_1+z_3^{''}\right] \cup$$
$$\cup \left[\widetilde{a}_1+\widetilde{e}_1+z_2^{''}, \widetilde{b}_1+\widetilde{f}_1+z_2^{''}\right] \cup \left[\widetilde{a}_1+z_2^{''}+z_3^{''}, \widetilde{b}_1+z_2^{''}+z_3^{''}\right] \cup \left[\widetilde{c}_1+z_1^{''}+z_3^{''}, \widetilde{d}_1+z_1^{''}+z_3^{''}\right] \cup$$
$$\cup \left[\widetilde{c}_1+\widetilde{e}_1+z_1^{''}, \widetilde{d}_1+\widetilde{f}_1+z_1^{''}\right] \cup \left[\widetilde{e}_1+z_1^{''}+z_2^{''}, \widetilde{f}_1+z_1^{''}+z_2^{''}\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *consists of no more one point: or* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \emptyset$, *or* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right) = \left\{z_1^{''}+z_2^{''}+z_3^{''}\right\}$, *where* $z_1^{''}$, $z_2^{''}$, *and* $z_3^{''}$ *are the eigenvalues of operators* $\widetilde{H}^1_{2\Lambda_1}$, $\widetilde{H}^2_{2\Lambda_2}$, *and* $\widetilde{H}^3_{2\Lambda_3}$, *correspondingly.*

f). Let $U > 0$, $\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} \leq U < \frac{6B}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, $\frac{6B}{W} \leq U < \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} < U \leq \frac{6B}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of four segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[\widetilde{a}_1+\widetilde{c}_1+\widetilde{e}_1, \widetilde{b}_1+\widetilde{d}_1+\widetilde{f}_1\right] \cup \left[\widetilde{a}_1+\widetilde{c}_1+z_3^{''}, \widetilde{b}_1+\widetilde{d}_1+z_3^{''}\right] \cup$$
$$\cup \left[\widetilde{c}_1+\widetilde{e}_1+z_1^{''}, \widetilde{d}_1+\widetilde{f}_1+z_1^{''}\right] \cup \left[\widetilde{c}_1+z_1^{''}+z_3^{''}, \widetilde{d}_1+z_1^{''}+z_3^{''}\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right) = \left[\widetilde{a}_1+\widetilde{c}_1+\widetilde{e}_1, \widetilde{b}_1+\widetilde{d}_1+\widetilde{f}_1\right] \cup \left[\widetilde{a}_1+\widetilde{c}_1+z_3^{''}, \widetilde{b}_1+\widetilde{d}_1+z_3^{''}\right] \cup$$
$$\cup \left[\widetilde{a}_1+\widetilde{e}_1+z_2^{''}, \widetilde{b}_1+\widetilde{f}_1+z_2^{''}\right] \cup \left[\widetilde{a}_1+z_2^{''}+z_3^{''}, \widetilde{b}_1+z_2^{''}+z_3^{''}\right].$$

The discrete spectrum of the the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\emptyset$.

k). Let $U>0$, $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}<U\leq\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}$, $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U>0$, $\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}<U\leq\frac{6B}{W}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U>0$, $\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}<U\leq\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$. *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is the union of two segments,*

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[\widetilde{a}_1+\widetilde{c}_1+\widetilde{e}_1,\widetilde{b}_1+\widetilde{d}_1+\widetilde{f}_1\right]\cup\left[\widetilde{c}_1+\widetilde{e}_1+z_1^{''},\widetilde{d}_1+\widetilde{f}_1+z_1^{''}\right],$$

or

$$\sigma_{ess}\left({}^4\widetilde{H}^q_{3/2}\right)=\left[\widetilde{a}_1+\widetilde{c}_1+\widetilde{e}_1,\widetilde{b}_1+\widetilde{d}_1+\widetilde{f}_1\right]\cup\left[\widetilde{a}_1+\widetilde{c}_1+z_3^{''},\widetilde{b}_1+\widetilde{d}_1+z_3^{''}\right].$$

The discrete spectrum of the operator ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\emptyset$.

l). Let $U>0$, $0<U\leq\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ *and* $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}$, $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, *or* $U>0$, $0<U\leq\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}$, ($\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}$). *Then the essential spectrum of the system fourth five-electron quartet state operator* ${}^4\widetilde{H}^q_{3/2}$ *is consists of a single segment:* $\sigma_{ess}({}^4\widetilde{H}^q_{3/2})=[\widetilde{a}_1+\widetilde{c}_1+\widetilde{e}_1,\widetilde{b}_1+\widetilde{d}_1+\widetilde{f}_1]$. *The discrete spectrum of the operator* ${}^4\widetilde{H}^q_{3/2}$ *is a empty set:* $\sigma_{disc}\left({}^4\widetilde{H}^q_{3/2}\right)=\emptyset$.

P r o o f. The proof of Theorems 13–14 is similar to the proof of Theorem 12. □

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Ташпулатов С. М. Структура существенного спектра и дискретный спектр оператора энергии пятиэлектронных систем в модели Хаббарда. Четвертое квартетное состояние. *Дальневосточный математический журнал.* 2023. Т. 23. № 1. С. 112–133.

АННОТАЦИЯ

Рассматривается оператор энергии пятиэлектронных систем в модели Хаббарда, исследуется структура существенного спектра и дискретный спектр системы в четвертом квартетном состоянии системы. Показано, что существенный спектр системы в четвертом квартетном состоянии является объединением не более чем семи отрезков, а дискретный спектр состоит из не более чем одной точки.

Ключевые слова: *пятиэлектронная система, модель Хаббарда, квартетное состояние, дублетное состояние, секстетное состояние, существенный спектр, дискретный спектр.*