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Computer design of cylindrical cloaking shell for the magnetostatics model

The problem of designing multilayer cylindrical radially anisotropic and isotropic shielding cloaks is investigated. Using the optimization method the design problem is reduced to finite-dimensional extremum problem, for which an efficient numerical algorithm based on the particle swarm optimization method is developed. Computational experiments have shown that the proposed method allows to design multilayer shielding cloak which has high performance and simplicity of technical implementation.

Key words: *optimization method, magnetic permeability, multilayer design, particle swarm optimization, shielding cloak.*

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Introduction

The problems of magnetic cloaking have received great development in recent years. It is related with important technological applications in biomedical processes and magnetically sensitive devices.

The first works in this field [1] were associated with using the tranformation optics (TO) method developed in the pioneering works [2, 3]. We emphasize that applying the TO method leads to singular solutions that are difficult to implement in practice.

Later, a group of authors [4,5] proposed another scheme for designing magnetic cloaking devices which is free from the mentioned drawback but it provides only approximate cloaking effect.

Another way to overcome mentioned drawbacks is to use the inverse design method based on the optimization method of solving inverse problems [6]. In a series of our own papers [7–13], we use this method for theoretical and numerical studies of different types of cloaking problems.

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In this paper, we use the inverse design method to solve the problem of designing a magnetic shielding cloak consisting of a finite number of layers filled with anisotropic or isotropic materials. Based on computational experiments, we will show that our algorithm allows to design a highly efficient shielding cloak that has a simple technical implementation.

1 Statement of direct and inverse problems of magnetostatics on a plane

Let us first formulate the direct problem of magnetostatics considered in the entire plane $\mathbb{R}^2$ filled with a homogeneous medium with constant magnetic permeability $\mu_0 > 0$. We assume that $\mathbb{R}^2$ has a constant magnetic field $\mathbf{H}_a = -\nabla\Phi_a$ corresponding to the magnetic potential $\Phi_a(\mathbf{x}) = -(H_a^0 r/b)\cos\varphi$, where $H_a^0 = \text{const}$, r, φ are the polar coordinates of the point $\mathbf{x} \in \mathbb{R}^2$.

We consider a physical scenario when an object (Ω, μ) is placed into the plane, where Ω is a ring shell $a < r < b$, and μ is a magnetic permeability of medium filling the domain Ω. Then the field Φ_a changes and takes the form $\Phi = \Phi_a + \Phi_s$, where Φ_s is perturbation of the field Φ_a caused by the placing of object into $\mathbb{R}^2$. To find the scattered response Φ_s it is necessary to formulate the direct problem of magnetostatics corresponding to the physical scenario described above.

We introduce two sets: $\Omega_0 = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < a\}$ and $\Omega_{M+1} = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| > b\}$ and replace the "continuous" shell Ω by a multilayer shell consisting of a finite number M of layers

$$\Omega_m = \{R_{m-1} < r = |\mathbf{x}| < R_m,\ m = \overline{1, M}\},\ \ R_0 = a,\ \ R_M = b,$$

of the same width $d = (b - a)/M$. Each of them is filled with a homogeneous anisotropic (generally) medium, whose constant magnetic permeability μ_m, $m = \overline{1, M}$, is described by the diagonal tensor in polar coordinates $\mu_m = \operatorname{diag}(\mu_{rm}, \mu_{\varphi m})$, where μ_{rm} (or $\mu_{\varphi m}$) is the radial (or tangential) component of the tensor μ_m . In what follows, to describe a piecewise homogeneous medium filling Ω, we will use the $2M$-dimensional vector $\mathbf{m} = (\mu_{r1},\ \mu_{\varphi 1},\ \ldots,\ \mu_{rM},\ \mu_{\varphi M})$, composed of the magnetic permeability components of all layers Ω_m, $m = \overline{1, M}$, and the pair $(\Omega, \mathbf{m})$ will be referred to as a magnetic shell.

Denote by Φ_m the restriction $\Phi|_{\Omega_m}$ of the total field $\Phi = \Phi_a + \Phi_s$ to the subdomain Ω_m, $m = \overline{0, M+1}$. Then the direct problem of finding the total field $\Phi = \Phi_a + \Phi_s$ reduces to finding all $M+2$ fields Φ_m in the domains Ω_m, $m = \overline{0, M+1}$, by solving the following magnetic conjugation problem:

$$\Delta\Phi_0 = 0 \text{ in } \Omega_0, \quad \operatorname{div}(\mu_m \nabla\Phi_m) = 0 \text{ in } \Omega_m,\ m = \overline{1, M}, \quad \Delta\Phi_{M+1} = 0 \text{ in } \Omega_{M+1}, \tag{1}$$

$$\Phi_m = \Phi_{m+1},\ \ \mu_{rm}\frac{\partial\Phi_m}{\partial r} = \mu_{r(m+1)}\frac{\partial\Phi_{m+1}}{\partial r} \ \text{ on } r = R_m,\ m = \overline{0, M}, \tag{2}$$

$$\Phi_0(\mathbf{x}) = O(1) \text{ as } r = |\mathbf{x}| \to 0, \quad \Phi_{M+1}(\mathbf{x}) \to \Phi_a(\mathbf{x}) \text{ as } r \to \infty, \tag{3}$$

considered in the entire plane $\mathbb{R}^2$. In (2) $\mu_{r0} = \mu_{r(M+1)} = \mu_0$ is constant magnetic permeability of homogeneous isotropic medium filling domains Ω_0 and Ω_{M+1}.

Similarly to works [14,15] devoted to solving related problems of electrostatic cloaking, we represent the fields Φ_0, Φ_1, ..., Φ_M, Φ_{M+1} as

$$\Phi_0(r,\varphi) = (A_0 r/b)\cos\varphi \text{ in } \Omega_0, \tag{4}$$

$$\Phi_m(r,\varphi) = ((r/b)^{\gamma_m} A_m + (b/r)^{\gamma_m} B_m)\cos\varphi \text{ in } \Omega_m, \quad m = \overline{1,M}, \tag{5}$$

$$\Phi_{M+1}(r,\varphi) = (-H_a^0 r/b + B_{M+1} b/r)\cos\varphi \text{ in } \Omega_{M+1}. \tag{6}$$

Here A_0, A_m, B_m, B_{M+1}, $m = \overline{1,M}$ are unknown coefficients, and γ_m is the coefficient (degree) of medium anisotropy in Ω_m, defined by the formula $\gamma_m{=}\sqrt{\mu_{\varphi m}/\mu_{rm}}$, $m{=}\overline{1,M}$. It is easy to check that all the equations in (1) and conditions (3) are satisfied for functions Φ_m, $m = \overline{0,M+1}$, defined in (4)–(6) for any values of the coefficients A_0, A_m, B_m, B_{M+1}, $m = \overline{1,M}$. It remains to choose them so that the boundary conditions (2) are satisfied. Substituting (4)–(6) into (2), we get:

$$\begin{gathered}
-A_0 + A_1 c_0^{1-\gamma_1} + B_1 c_0^{1+\gamma_1} = 0,\\
-\mu_0 A_0 + \mu_{r1} A_1 \gamma_1 c_0^{1-\gamma_1} - \mu_{r1} B_1 \gamma_1 c_0^{1+\gamma_1} = 0,\\
-A_m c_m^{-\gamma_m} - B_m c_m^{\gamma_m} + A_{m+1} c_m^{-\gamma_{m+1}} + B_{m+1} c_m^{\gamma_{m+1}}{=}0,\\
-\mu_{rm} A_m \gamma_m c_m^{1-\gamma_m} + \mu_{rm} B_m \gamma_m c_m^{1+\gamma_m} + \mu_{r(m+1)} A_{m+1} \gamma_{m+1} c_m^{1-\gamma_{m+1}} -\\
-\mu_{r(m+1)} B_{m+1} \gamma_{m+1} c_m^{1+\gamma_{m+1}} = 0, \quad m = \overline{1,M-1},\\
-A_M - B_M + B_{M+1} = H_a^0,\\
-\mu_{rM} A_M \gamma_M + \mu_{rM} B_M \gamma_M - \mu_{M+1} B_{M+1} = \mu_{M+1} H_a^0.
\end{gathered} \tag{7}$$

Here $c_m = b/R_m$, $m = \overline{0,M-1}$. Equalities (7) are a system of $2M+2$ linear algebraic equations with respect to $2M+2$ unknown coefficients A_0, A_m, B_m, B_{M+1}, $m = \overline{1,M}$, which must be solved to find the fields Φ_m. It is easy to establish that the system matrix (7) is non-singular, except for some special values of magnetic permeability. Therefore, the system (7) can be solved with respect to the unknown coefficients.

Below we will also consider a simpler particular case of the system (7), corresponding to the anisotropy coefficients $\gamma_m = 1$, $m = \overline{1,M}$. Such a case corresponds to the design of a completely isotropic multilayer shell $(\Omega, \mathbf{m})$. Here $\mathbf{m} = (\mu_1, \mu_2, ..., \mu_M)$ where μ_m are the constant magnetic permeabilities of homogeneous isotropic layers Ω_m, $m = \overline{1,M}$.

Denote further by $\Phi[\mathbf{m}]{=}(\Phi_0[\mathbf{m}], \Phi_1[\mathbf{m}], ..., \Phi_{M+1}[\mathbf{m}])$, where $\mathbf{m} = (\mu_{r1}, \mu_{\varphi 1}, ..., \mu_{rM}, \mu_{\varphi M})$, solution of problem (1)-(3) corresponding to the tensor magnetic permeability μ_m in Ω_m, $m = \overline{1,M}$, and the constant magnetic permeability μ_0 in Ω_0 and Ω_{M+1}. We put $\Omega_e = \Omega_{M+1} \cap B_R$, where B_R is a circle of large radius R containing Ω inside it.

Below we will consider an inverse problem called the magnetic shielding problem [10, 11]. It consists in finding the magnetic permeability vector $\mathbf{m} = (\mu_{r1}, \mu_{\varphi 1}, ..., \mu_{rM}, \mu_{\varphi M})$, based on the following condition: $\nabla\Phi_0[\mathbf{m}] = 0$, i.e. $\Phi_0[\mathbf{m}] = \text{const}$ in Ω_0.

2 Statement of the control problem

To solve the inverse problem, we apply the optimization method [6]. Following it, we define the cost functional:

$$J_i(\mathbf{m})=\frac{\|\nabla\Phi_i[\mathbf{m}]\|_{L^2(\Omega_0)}}{\|\nabla\Phi_a\|_{L^2(\Omega_0)}},\ \|\nabla\Phi_a\|^2_{L^2(\Omega_0)}=\int\limits_{\Omega_0}|\nabla\Phi_a|^2 d\mathbf{x},\ \|\nabla\Phi_i[\mathbf{m}]\|^2_{L^2(\Omega_0)}=\int\limits_{\Omega_0}|\nabla\Phi_i[\mathbf{m}]|^2 d\mathbf{x},$$

and the following bounded set called the control set:

$$K=\{\mathbf{m}=(\mu_{r1},\mu_{\varphi 1},...,\mu_{rM},\mu_{\varphi M})\in\mathbb{R}^{2M}:0<\mu_{min}\le(\mu_{rm},\mu_{\varphi m})\le\mu_{max}\}. \tag{8}$$

Here, the given positive constants μ_{min} and μ_{max} determine the lower and upper bounds of the set K. Let us formulate the extremum shielding problem having the form

$$J_i(\mathbf{m})\to\inf,\quad \mathbf{m}\in K. \tag{9}$$

We note that the value $J_i(\mathbf{m})$ is associated with shielding performance of $(\Omega,\mathbf{m})$ via inverse dependence: the smaller value $J_i(\mathbf{m})$ corresponds to the higher shielding performance of $(\Omega,\mathbf{m})$, and vice versa [7,10–13]. Recall that there exists a solution of the problem (9) while the set K is closed and bounded in the space $\mathbb{R}^{2M}$ and the function $J_i(\mathbf{m})$ is continuous on K [13, ch. 6]. It remains to find this solution using some algorithm.

3 Analysis of computational experiments results

Let us discuss here the results of numerical solution of the problem (9) using the particle swarm optimization method (PSO) [16] for the initial data: $a=0.04$ m, $b=0.05$ m, $\mu_0=1$, $R=3$ m, and two pairs of values $(0.08;20)$ and $(1;5000)$ defining the boundaries μ_{min} and μ_{max} of control set K in (8).

Our first test is related with solving the extremum problem (9) using the PSO for a fully anisotropic multilayer shell $(\Omega,\mathbf{m})$ for the first pair $\mu_{min}=0.08$ and $\mu_{max}=20$.

Optimization analysis using PSO for different values of M led to the results presented in Table 1 in the form of optimal values of the radial and tangential components μ_{rm}^{opt} and $\mu_{\varphi m}^{opt}$ of the magnetic permeabilities of layers (Ω_m,μ_m) and the corresponding minimum value $J_i(\mathbf{m}^{opt})$ of the functional J_i, where $\mathbf{m}^{opt}=(\mu_{r1}^{opt},\mu_{\varphi 1}^{opt},...,\mu_{rM}^{opt},\mu_{\varphi M}^{opt})$.

From Table 1, it can be seen, in particular, that the optimal values of the magnetic permeabilities of all layers found using the PSO method coincide in each layer for any $M=\overline{1,12}$, i.e. all layers are filled with the same anisotropic medium with permeabilities $\mu_r^{opt}=0.08$ and $\mu_\varphi^{opt}=20$. At the same time the values $J_i(\mathbf{m}^{opt})$ are equal to 3.62×10^{-2} and do not change with increasing M, where the latter value corresponds to the weak shielding performance of the designed anisotropic shielding cloak. Thereby, from Table 1 it follows that solving the problem (9) in the case of a completely anisotropic shell for the first pair of parameters $\mu_{min}=0.08$ and $\mu_{max}=20$ fails to achieve neither high shielding performance due to the multilayer design of shielding cloak, nor its easy technical implementation due to the anisotropy $\gamma_m=15.8$ of the layers.

Table 1: Shielding problem: μ_{min}=0.08, μ_{max}=20, contrast=μ_{max}/μ_{min}=250, γ_m=15.8

M	$(\mu_{r1}^{opt}, \mu_{\varphi 1}^{opt})$	$(\mu_{r2}^{opt}, \mu_{\varphi 2}^{opt})$	...	$(\mu_{rM}^{opt}, \mu_{\varphi M}^{opt})$	$J_i(\mathbf{m}^{opt})$
1	$(0.08, 20)$				$3.62{\cdot}10^{-2}$
2	$(0.08, 20)$	$(0.08, 20)$			$3.62{\cdot}10^{-2}$
4	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$3.62{\cdot}10^{-2}$
6	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$3.62{\cdot}10^{-2}$
8	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$3.62{\cdot}10^{-2}$
10	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$3.62{\cdot}10^{-2}$
12	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$(0.08, 20)$	$3.62{\cdot}10^{-2}$

For simplicity of implementation, all layers of the designed shielding cloak must be filled with isotropic media (i.e. $\gamma_m = 1$), which correspond to available natural materials. Therefore, our next test 2 is related with solution of the problem (9) using PSO for the case of a completely isotropic multilayer shielding cloak $(\Omega, \mathbf{m})$, where $\gamma_m = 1$, for the second pair of parameters $\mu_{min} = 1$ and μ_{max} =5000. Recall that the value $\mu_{min} = 1$ describes with great accuracy the magnetic permeability of air, wood and other natural materials, while the value $\mu_{max} = 5000$ describes the permeability of iron.

Applying PSO for solving (9) leads to the results listed in Table 2 as the optimal values of the constant permeabilities μ_1^{opt}, μ_2^{opt}, μ_{M-1}^{opt}, μ_M^{opt} and the corresponding minimum value $J_i(\mathbf{m}^{opt})$ of functional J_i for even $M = \overline{2, 12}$.

Table 2: Shielding problem: μ_{min}=1, μ_{max}=5000, contrast=μ_{max}/μ_{min}=5000, γ_m=1

M	μ_1^{opt}	μ_2^{opt}	μ_{M-1}^{opt}	μ_M^{opt}	$J_i(\mathbf{m}^{opt})$
2	5000	5000			$2.22{\cdot}10^{-3}$
4	5000	1	5000	5000	$2.49{\cdot}10^{-4}$
6	5000	1	5000	5000	$8.57{\cdot}10^{-5}$
8	5000	1	5000	5000	$4.83{\cdot}10^{-5}$
10	5000	1	5000	5000	$3.49{\cdot}10^{-5}$
12	5000	1	5000	5000	$2.87{\cdot}10^{-5}$

The results shown in Table 2 demonstrate that the obtained optimal solution $\mathbf{m}^{opt}$, up to the last permeability, obeys to an analogue of the bang-bang principle in the sense that the following relation holds

$$\mu_1^{opt} = \mu_3^{opt} = ... = \mu_{M-1}^{opt} = \mu_{max}, \quad \mu_2^{opt} = \mu_4^{opt} = ... = \mu_{M-2}^{opt} = \mu_{min},$$

called in [10,11] as the alternating design scheme (see [12,13] for more details), while the magnetic permeability μ_M^{opt} in the last layer always takes the value of $\mu_{max} = 5000$.

It follows from Table 2 that for the optimal solution $\mathbf{m}^{opt}$ of the shielding problem (9) in the test 2 the value $J_i(\mathbf{m}^{opt})$ of functional J_i decreases from value $2.22 \cdot 10^{-3}$ for

$M = 2$ to value $2.87 \cdot 10^{-5}$ for $M = 12$ which corresponds to very high shielding performance of the respective shield $(\Omega, \mathbf{m}^{opt})$. Moreover, all optimal permeabilities obtained in Table 2 corresponds only to two readily available natural materials. Therefore, the designed shielding cloak is technically easily implemented.

In conclusion, the optimization analysis showed that a high performance of the designed magnetic shielding device can be achieved when using multilayer cloak consisting of several isotropic homogeneous layers with optimal constant permeabilities obtained by the developed numerical algorithm based on PSO. We emphasize that high shielding performance and simplicity of technical realization can be achieved without use of anisotropic methamaterials, but using only two natural materials with high contrast.

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Спивак Ю. Э. Компьютерный дизайн цилиндрической маскировочной оболочки для модели магнитостатики. *Дальневосточный математический журнал.* 2022. Т. 22. № 2. С. 238–244.

АННОТАЦИЯ

Исследуется задача дизайна многослойных цилиндрических радиально-анизотропных и изотропных экранирующих оболочек. С использованием оптимизационного метода задача дизайна сводится к конечномерной экстремальной задаче, для решения которой разрабатывается эффективный численный алгоритм, основанный на методе роя частиц. Вычислительные эксперименты показали, что предложенный метод позволяет спроектировать многослойную экранирующую оболочку, обладающую высокой эффективностью и простотой технической реализации.

Ключевые слова: *оптимизационный метод, магнитная проницаемость, многослойный дизайн, метод роя частиц, экранирующая оболочка.*