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Inverse problem of recovering the electron diffusion coefficient

The inverse problem of recovering the electron diffusion coefficient is considered. Within the framework of the optimization approach, this problem is reduced to the multiplicative control one. The solvability of the considered extremum problem is proven.

Key words: *drift-diffusion electron model, polar dielectric charging model, multiplicative control problem, inverse coefficients problem.*

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Introduction. Statement of boundary value problem

In recent years, there has been an interest in the study of the diffusion-drift approach for modeling the charging process of polar dielectrics induced by electron irradiation. On the practical side, this area is of interest due to the need to predict the state of dielectric materials under their diagnosis by scanning electron microscopy methods.

A mathematical model of the charging process of polar dielectrics with sufficiently long electron irradiation can be represented by the following boundary value problem considered in a bounded domain $\Omega \subset \mathbb{R}^3$ with a boundary Γ:

$$-\mathrm{div}\,(d\,\nabla\rho) + \mu_n \mathbf{E}\cdot\nabla\rho + (\mu_n/\varepsilon\varepsilon_0)|\rho|\rho = f \text{ in } \Omega, \tag{1}$$

$$\mathrm{curl}\,\mathbf{E} = \mathbf{0}, \quad \mathrm{div}\,\mathbf{E} = (1/\varepsilon\varepsilon_0)\rho \text{ in } \Omega, \tag{2}$$

$$\rho = 0, \;\; \mathbf{E}\times\mathbf{n} = \mathbf{0} \text{ on } \Gamma. \tag{3}$$

Here ρ is the volume charge density, $\mathbf{E}$ is the electric field intensity vector, $d(\mathbf{x})$ is the diffusion coefficient of electrons, μ_n is the drift mobility of electrons, ε is the dielectric permittivity, ε_0 is the dielectric constant, f is the generating term responsible for the action of a volume charge source in an object. Below we will refer to the problem (1)–(3) as Problem 1.

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A wide range of modern works is devoted to the development of mathematical models and the creation of software for the study of electronically stimulated charging processes (see, for example, [1–6]). In turn, the mathematical correctness of one of these models was established recently in the article [7], where the global solvability of Problem 1 and local uniqueness of its solution are proved. For the charge density ρ, the maximum principle was established. Also, the maximum principle was applied to the control of computational experiments.

In this paper, the multiplicative control problem for Problem 1 is formulated and its solvability is proved. The control role is played by the function d. Note that, within the framework of the optimization approach, the problem of recovering the coefficient d, based on additional information about the solution of Problem 1, can be reduced to the control problem under consideration (see [8–10]).

1 Solvability of the boundary value problem

When analyzing the boundary value problem, we will use Sobolev functional spaces $H^s(D)$, $s \in \mathbb{R}$. Here, D denotes domain Ω, or some subdomain $Q \subset \Omega$, or boundary Γ. By $\|\cdot\|_{s,Q}$, $|\cdot|_{s,Q}$, and $(\cdot,\cdot)_{s,Q}$, we denote the norm, semi-norm, and scalar product in $H^s(Q)$. Norms and scalar products in $L^2(Q)$ and $L^2(\Omega)$ we denote respectively by $\|\cdot\|_Q$ and $(\cdot,\cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot,\cdot)_\Omega$.

We introduce the following function spaces $H^1(\Delta,\Omega) = \{h \in H^1(\Omega) : \Delta h \in L^2(\Omega)\}$, $H^1_N(\Omega) = \{\mathbf{h} \in H^1(\Omega)^3 : \mathbf{h} \times \mathbf{n}|_\Gamma = \mathbf{0}\}$, $\widetilde{H}^1_N(\Omega) = H^1_N(\Omega) \cap \ker(\mathrm{curl})$, the function set $H^s_{d_0}(\Omega) = \{d \in H^s(\Omega) : d \geq d_0 > 0\}$, $s > 3/2$, and the space $X = H^1_0(\Omega) \times \widetilde{H}^1_N(\Omega)$.

Let the following conditions hold:

(i) Ω is a bounded simply connected domain in $\mathbb{R}^3$ with connected boundary $\Gamma \in C^{0,1}$;

(ii) $f \in L^2(\Omega)$, $d \in H^s_{d_0}(\Omega)$.

Let us also remind that by the Sobolev embedding theorem, the space $H^1(\Omega)$ is embedded into space $L^s(\Omega)$ continuously for $s \leq 6$, compactly for $s < 6$, and the following estimate holds:

$$\|h\|_{L^s(\Omega)} \leq C_s \|h\|_{1,\Omega} \quad \forall h \in H^1(\Omega), \tag{4}$$

where the constant C_s depends only on s and Ω.

The following lemmas hold (see [11]).

Lemma 1. *Under the conditions (i) and $\mathbf{E} \in H^1(\Omega)^3$, there exist positive constants C_0, δ_1, γ_1', and γ_1, which depend on Ω such that*

$$|(\nabla h, \nabla \eta)_\Omega| \leq C_0 \|h\|_{1,\Omega} \|\eta\|_{1,\Omega},$$

$$|(\mathbf{E} \cdot \nabla h, \eta)| \leq \gamma_1' \|\mathbf{E}\|_{L^4(\Omega)^3} \|h\|_{1,\Omega} \|\eta\|_{1,\Omega} \leq \gamma_1 \|\mathbf{E}\|_{1,\Omega} \|h\|_{1,\Omega} \|\eta\|_{1,\Omega} \;\; \forall h, \eta \in H^1(\Omega), \tag{5}$$

$$(\nabla s, \nabla s) \geq \delta_1 \|s\|^2_{1,\Omega} \;\; \forall s \in H^1_0(\Omega). \tag{6}$$

If the functions $\mathbf{E} \in H^1(\Omega)^3$ and $\rho \in H^1_0(\Omega)$ are related by the second relation in (2), then the following relation $(\mathbf{E} \cdot \nabla \rho, h) = -(\nabla h \cdot \mathbf{E}, \rho) - (1/\varepsilon\varepsilon_0)(h, \rho^2)$ takes place for all $h \in H^1_0(\Omega)$. If $h = \rho$, it takes the following form: $\mu_n(\mathbf{E} \cdot \nabla \rho, \rho) = -(\mu_n/2\varepsilon\varepsilon_0)(\rho, \rho^2)$.

Lemma 2. *Under the condition (i) for any function* $\sigma \in L^2(\Omega)$*, there is a unique solution* $\mathbf{E} \in \widetilde{H}^1_N(\Omega)$ *of the problem:* $\mathrm{curl}\,\mathbf{E} = \mathbf{0}$*,* $\mathrm{div}\,\mathbf{E} = \sigma$ *in* Ω*,* $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ *on* Γ *and the following estimate holds:* $\|\mathbf{E}\|_{1,\Omega} \leq C_N \|\sigma\|_\Omega$*, where* C_N *is a positive constant which depends only on* Ω*.*

Let $(\rho, \mathbf{E}) \in (C^2(\Omega) \cap C^0(\overline{\Omega})) \times (C^1(\Omega)^3 \cap \widetilde{H}^1_N(\Omega))$ is the classical solution of Problem 1. Let us multiply the equation in (1) by a function $h \in H^1_0(\Omega)$ and integrate over Ω using the Green's formula. As a result, we obtain the weak formulation of Problem 1:

$$(d\,\nabla\rho, \nabla h) + \mu_n(\mathbf{E} \cdot \nabla\rho, h) + (\mu_n/\varepsilon\varepsilon_0)(|\rho|\rho, h) = (f, h) \quad \forall h \in H^1_0(\Omega), \tag{7}$$

$$\mathrm{div}\,\mathbf{E} = (1/\varepsilon\varepsilon_0)\rho \text{ in } \Omega. \tag{8}$$

Theorem 1. *Assume that the assumptions (i), (ii) hold. Then there exists a weak solution* $(\rho, \mathbf{E}) \in X$ *of Problem 1 and the following estimates are valid:*

$$\|\rho\|_{1,\Omega} \leq C_* \|f\|_\Omega, \quad \|\mathbf{E}\|_{1,\Omega} \leq (1/\varepsilon\varepsilon_0) C_2 C_N C_* \|f\|_\Omega, \quad C_* = (d_0\delta_1)^{-1}.$$

Besides, if the condition $(\mu_n C_2 C_4/\varepsilon\varepsilon_0)(\gamma_1 C_N + C_4)\|f\|_\Omega < \lambda_*^2$*, where* $\lambda_* = d_0\delta_1$*, holds, then the weak solution of Problem 1 is unique.*

2 Statement and solvability of the control problem

In this section we will study a multiplicative control problem for the system (1)–(3), in which the role of the control is played by coefficient d. We assume that the function d can be changed in K which satisfies the following condition:

(j) $K \subset H^s_{d_0}(\Omega)$, $s > 3/2$, is a nonempty convex closed set.

Let us introduce the functional spaces $X = H^1_0(\Omega) \times \widetilde{H}^1_N(\Omega)$, $Y = H^{-1}(\Omega) \times \widetilde{H}^1_N(\Omega)^*$ and set $\mathbf{x} = (\rho, \mathbf{E}) \in X$. Further, we consider the operator $F = (F_1, F_2) : X \times K \to Y$ by formulae $\langle F_1(\mathbf{x}, d), h\rangle = (d\,\nabla\rho, \nabla h) + \mu_n(\mathbf{E} \cdot \nabla\rho, h) + (\mu_n/\varepsilon\varepsilon_0)(|\rho|\rho, h) - (f, h)$, $F_2(\mathbf{x}) = \mathrm{div}\,\mathbf{E} - (1/\varepsilon\varepsilon_0)\rho$ and rewrite a weak form (7)–(8) of Problem 1 in the form of the operator equation $F(\mathbf{x}, d) = 0$.

Let $I : X \to \mathbb{R}$ be a weakly lower semicontinuous functional. We consider the following multiplicative control problem:

$$J(\mathbf{x}, d) \equiv (\mu_0/2)I(\mathbf{x}) + (\mu_1/2)\|d\|^2_{s,\Omega} \to \inf, \quad F(\mathbf{x}, d) = 0, \quad (\mathbf{x}, d) \in X \times K. \tag{9}$$

The set of possible pairs for the problem (9) is denoted by $Z_{ad} = \{(\mathbf{x}, d) \in X \times K : F(\mathbf{x}, d) = 0,\ J(\mathbf{x}, d) < \infty\}$.

Let, in addition to (j), the following condition hold:

(jj) $\mu_0 > 0$, $\mu_1 \geq 0$, and K is a bounded set in $H^s(\Omega)$, $s > 3/2$, or $\mu_i > 0$, $i = 0, 1$, and the functional I is bounded from below.

We use the following cost functionals:

$$I_1(\rho) = \|\rho - \rho^d\|^2_Q, \quad I_2(\mathbf{E}) = \|\mathbf{E} - \mathbf{E}^d\|^2_Q. \tag{10}$$

Here, the function $\rho^d \in L^2(Q)$ denotes a desired volume charge density in a subdomain $Q \subset \Omega$. The function $\mathbf{E}^d$ has a similar sense as the electric field.

Theorem 2. *Assume that the assumptions (i), (ii) and (j), (jj) take place. Let* $I : X \to \mathbb{R}$ *be a weakly semicontinuous below functional and let* $Z_{ad} \neq \emptyset$. *Then there is at least one solution* $(\mathbf{x}, d) \in X \times K$ *of the control problem* (9).

P r o o f. Let $(\mathbf{x}_m, d_m) = (\rho_m, \mathbf{E}_m, d_m) \in Z_{ad}$ is a minimizing sequence for which the following is true: $\lim_{m\to\infty} J(\mathbf{x}_m, d_m) = \inf_{(\mathbf{x},d)\in Z_{ad}} J(\mathbf{x}, d) \equiv J^*$.

From the condition (jj) and from the Theorem 1, it can be deduced that the following estimates hold:

$$\|d_m\|_{s,\Omega} \le c_1, \quad \|\rho_m\|_{1,\Omega} \le c_2, \quad \|\mathbf{E}_m\|_{1,\Omega} \le c_3, \tag{11}$$

where the constants c_1, c_2, c_3 do not depend on m. From the estimate (11) and from the condition (j), it follows that there exist weak limits $d^* \in K$, $\rho^* \in H_0^1(\Omega)$, and $\mathbf{E}^* \in \widetilde{H}_N^1(\Omega)$ of some subsequences of sequences $\{d_m\}$, $\{\rho_m\}$, and $\{\mathbf{E}_m\}$, respectively.

With this in mind, it can be considered that as $m \to \infty$, we have

$$\begin{aligned} \rho_m \to \rho^* \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^s(\Omega), \quad s < 6,\\ \mathbf{E}_m \to \mathbf{E}^* \text{ weakly in } H^1(\Omega)^3 \text{ and strongly in } L^p(\Omega)^3, \quad p < 6,\\ d_m \to d^* \text{ weakly in } H^s(\Omega) \text{ and strongly in } L^\infty(\Omega), \quad s > 3/2. \end{aligned} \tag{12}$$

It is clear that $F_2(\mathbf{x}^*) = 0$. Let us show that $F_1(\mathbf{x}^*, d^*) = 0$, that is

$$(d^*\,\nabla\rho^*, \nabla h) + \mu_n(\mathbf{E}^* \cdot \nabla\rho^*, h) + (\mu_n/\varepsilon\varepsilon_0)(|\rho^*|\rho^*, h) = (f, h)\ \forall h \in H_0^1(\Omega).$$

We remind that a pair $(\mathbf{x}_m, d_m)$ satisfies the relation

$$(d_m\,\nabla\rho_m, \nabla h) + \mu_n(\mathbf{E}_m \cdot \nabla\rho_m, h) + (\mu_n/\varepsilon\varepsilon_0)(|\rho_m|\rho_m, h) = (f, h)\ \forall h \in H_0^1(\Omega). \tag{13}$$

Let us pass in (13) to the limit as $m \to \infty$, starting with the term $(d_m\nabla\rho_m, \nabla h)$:

$$(d_m\nabla\rho_m, \nabla h) - (d^*\nabla\rho^*, \nabla h) = ((d_m - d^*)\nabla\rho_m, \nabla h) + (\nabla(\rho_m - \rho^*), d^*\nabla h). \tag{14}$$

Since $d^*\nabla h \in L^2(\Omega)^3$, according to (12), we obtain that

$$(\nabla(\rho_m - \rho^*), d^*\nabla h) \to 0 \text{ as } m \to \infty\ \ \forall h \in H_0^1(\Omega).$$

Using Holder's inequality and considering (12) and (11), we have

$$|((d_m - d^*)\nabla\rho_m, \nabla h)| \le \|d_m - d^*\|_{L^\infty(\Omega)}\|\nabla\rho_m\|_\Omega\|\nabla h\|_\Omega \to 0 \ \text{ as } \ m \to \infty\ \ \forall h \in H_0^1(\Omega).$$

In that case $(d_m\nabla\rho_m, \nabla h) \to (d^*\nabla\rho^*, \nabla h)$ as $m \to \infty$ for all $h \in H_0^1(\Omega)$.

It is clear that

$$(\mathbf{E}_m \cdot \nabla\rho_m, h) - (\mathbf{E}^* \cdot \nabla\rho^*, h) = ((\mathbf{E}_m - \mathbf{E}^*) \cdot \nabla\rho_m, h) + (\mathbf{E}^* \cdot \nabla(\rho_m - \rho^*), h).$$

By (12), using Lemma 1 and (11), for the first term we obtain that

$$\big|((\mathbf{E}_m - \mathbf{E}^*) \cdot \nabla\rho_m, h)\big| \le \gamma_1'\|\mathbf{E}_m - \mathbf{E}^*\|_{L^4(\Omega)^3}\|\rho_m\|_{1,\Omega}\|h\|_{1,\Omega} \to 0 \ \text{ as } \ m \to \infty\ \ \forall h \in H_0^1(\Omega).$$

Since $\mathbf{E}^* h \in L^2(\Omega)^3$, for the second term due to (12), we have

$$(\mathbf{E}^* \cdot \nabla(\rho_m - \rho^*), h) = (\nabla(\rho_m - \rho^*), \mathbf{E}^* h) \to 0 \text{ as } m \to \infty \ \forall h \in H_0^1(\Omega).$$

Thus, $(\mathbf{E}_m \cdot \nabla \rho_m, h) \to (\mathbf{E}^* \cdot \nabla \rho^*, h)$ as $m \to \infty$ for all $h \in H_0^1(\Omega)$.

Further, we consider the inequality

$$\left|\left(|\rho_m|\rho_m - |\rho^*|\rho^*, h\right)\right| \leq \left|\left(|\rho_m|(\rho_m - \rho^*), h\right)\right| + \left|\left(|\rho_m| - |\rho^*|, \rho^* h\right)\right|.$$

It is clear, that $(|\rho_m| - |\rho^*|, \rho^* h) \to 0$ as $m \to \infty$. Using Holder's inequality and taking into account (11) and (12) as $s = 4$, for the first term we obtain that

$$\left|\left(|\rho_m|(\rho_m - \rho^*), h\right)\right| \leq c_2 C_4 \|\rho_m - \rho^*\|_{L_4(\Omega)} \|h\|_{L^4(\Omega)} \to 0 \text{ as } m \to \infty \ \forall h \in H_0^1(\Omega).$$

Therefore $(|\rho_m|\rho_m, h) \to (|\rho^*|\rho^*, h)$ as $m \to \infty$ for all $h \in H_0^1(\Omega)$.

Since the functional J is weakly semicontinuous below on $X \times H^s(\Omega)$, we obtain from (11) that $J(\mathbf{x}^*, d^*) = J^*$.

□

Remark 1. It is clear, that all cost functionals from (10) satisfy the conditions of the Theorem 2.

References

[1] M. Kotera, K. Yamaguchi, H. Suga, "Dynamic Simulation of Electron-Beam-Induced Charging up of Insulators", *Jpn. J. Appl. Phys.*, **38**:12, (1999), 7176–7179.

[2] A. G. Maslovskaya, "Physical and mathematical modeling of the electron-beam-induced charging of ferroelectrics during the process of domain structure switching", *Jour. of Surface Investigation*, **7 (4)**, (2013), 680–684.

[3] A. V. Pavelchuk, A. G. Maslovskaya, "Approach to numerical implementation of the drift-diffusion model of field effects induced by a moving source", *Russ. Phys. J.*, **63**, (2020), 105–112.

[4] B. Raftari, N. V. Budko, C. Vuik, "Self-consistence drift-diffusion-reaction model for the electron beam interaction with dielectric samples", *J. Appl. Phys.*, **118**, (2015), 204101 (17).

[5] D. S. Chezganov, D. K. Kuznetsov, V. Ya. Shur, "Simulation of spatial distribution of electric field after electron beam irradiation of MgO-doped $LiNbO_3$ covered by resist layer", *Ferroelectrics*, **496**, (2016), 70–78.

[6] A. Maslovskaya, A. Pavelchuk, "Simulation of dynamic charging processes in ferroelectrics irradiated with SEM", *Ferroelectrics*, **476**, (2015), 157–167.

[7] R. V. Brizitskii, N. N. Maksimova, A. G. Maslovskaya, "Theoretical analysis and numerical implementation of a stationary diffusion-drift model for charging polar dielectrics", *Comp. Math. Math. Phys.*, **62**, (2022), (to appear).

[8] G. V. Alekseev, V. G. Romanov, "One class of nonscattering acoustic shells for a model of anisotropic acoustics", *J. Appl. Industr. Math.*, **6 (1)**, (2012), 1–5.

[9] G. V. Alekseev, V. A. Levin, D. A. Tereshko, "The optimization method in design problems of spherical layered thermal shells", *Doklady Physics.*, **62 (10)**, (2017), 465–469.

[10] R. V. Brizitskii, Zh. Yu. Saritskaya, “Optimization analysis of the inverse coefficient problem for the nonlinear convection-diffusion-reaction equation”, *Inverse Ill-Posed Probl.*, **26**:6, (2018), 821-833.

[11] G. V. Alekseev, “Optimization in stationary problems of heat and mass transfer and magnetic hydrodynamics.”, *M.: Nauch. Mir.*, 2010, 412.

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АННОТАЦИЯ

Рассмотрена обратная задача для определения коэффициента диффузии электронов по плотности заряда, измеренной в некотором фрагменте заряженного сегнетоэлектрика. В рамках оптимизационного подхода эта задача сводится к задаче мультипликативного управления. Доказана разрешимость рассматриваемой экстремальной задачи.

Ключевые слова: *дрейфово-диффузионная электронная модель, модель заряда полярного диэлектрика, задача мультипликативного управления, задача обратных коэффициентов.*