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On regular systems of algebraic p-adic numbers of arbitrary degree in small cylinders

In this paper we prove that for any sufficiently large $Q \in \mathbb{N}$ there exist cylinders $K \subset \mathbb{Q}_p$ with Haar measure $\mu(K) \leq \frac{1}{2}Q^{-1}$ which do not contain algebraic p-adic numbers α of degree $\deg \alpha = n$ and height $H(\alpha) \leq Q$. The main result establishes in any cylinder K, $\mu(K) > c_1 Q^{-1}$, $c_1 > c_0(n)$, the existence of at least $c_3 Q^{n+1}\mu(K)$ algebraic p-adic numbers $\alpha \in K$ of degree n and $H(\alpha) \leq Q$.

Key words: *integer polynomials, algebraic p-adic numbers, regular system, Haar measure.*

1 Introduction

The concept of a regular system of points is a convenient tool for the study of the uniform distribution of algebraic numbers. Regular systems were introduced by Baker and Schmidt [1] as a technique for obtaining a lower bound for the Hausdorff dimension of sets of real numbers close to infinitely many points of the set of algebraic numbers of bounded degree.

Definition 1. *Let Γ be a countable set of real numbers and let $N : \Gamma \to \mathbb{R}$ be a positive function. The pair (Γ, N) is called a regular system of points if there exists a constant $C = C(\Gamma, N) > 0$ such that for any finite interval I there exists a sufficiently large number $T_0 = T_0(\Gamma, N, I)$ such that for any integer $T \geq T_0$ there exists a collection $\gamma_1, \ldots, \gamma_{\mathbf{t}} \in \Gamma \cap I$ such that $N(\gamma_i) \leq T\,(1 \leq i \leq \mathbf{t})$, $|\gamma_i - \gamma_j| \geq T^{-1}\,(1 \leq i < j \leq \mathbf{t})$, and $\mathbf{t} \geq C|I|T$.*

Regular systems play the key role in the proof of the divergence case in the Khintchine-Groshev type theorems [2, 3, 4, 5] and obtaining lower bounds for the Hausdorff dimension of sets of number theoretic interest [1, 6, 7, 8, 9, 10].

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Y. Bugeaud in [11] stated the problem on finding an explicit dependence of T_0 on the length of the interval I. In [11] it is shown that for a given finite interval I in $[-1/2, 1/2]$ the value of $T_0(\Gamma, N, I)$ in the definition of regular system is equal to

$$T_0(\mathbb{Q}, N, I) = 10^4 |I|^{-2} \log^2 100|I|^{-1}$$

for $\Gamma = \mathbb{Q}$, and in [12] that

$$T_0(A_2, N, I) = 72^3 |I|^{-3} \log^3 72|I|^{-1}$$

for $\Gamma = A_2$, where A_n is the set of real algebraic numbers of degree n. Throughout $c_1 = c_1(n)$, $c_2 = c_2(n)$, ... are constants depending only on n. In [13] it is shown that $T_0(A_3, N, I) = c_1|I|^{-4-\epsilon}$, $0 < \epsilon < 1$. There is a more strong connection between I and $T_0(A_n, N, I)$, namely $T_0(A_n, N, I) = c_2|I|^{-(n+1)}$, see [14]. In this paper, we address the problem of Bugeaud for the p-adic algebraic numbers of arbitrary degree n.

The Haar measure of a measurable set $S \subset \mathbb{Q}_p$ is denoted by $\mu(S)$. Let $\mathcal{A}_p$ be the set of all algebraic numbers and $\mathbb{Q}_p^*$ be the extension of $\mathbb{Q}_p$ containing $\mathcal{A}_p$. The cylinder in $\mathbb{Q}_p$ of radius r centered at α is the set of solutions of the inequality $|w - \alpha|_p \leq r$. Denote by $\mathcal{A}_{n,p}$ the set of algebraic numbers of degree n lying in $\mathbb{Z}_p$. Fix any finite cylinder K_0 in $\mathbb{Z}_p$. The natural number $H(\alpha)$ denotes the naive height of $\alpha \in \mathcal{A}_p$, i.e. the maximum absolute value of the coefficients of the minimal integer polynomial of α. We will also use the Vinogradov symbol $f \ll g$ which means that there exists a constant $c > 0$ such that $f \leq cg$.

Theorem 1. *Let K be a finite cylinder in K_0. Then there are positive constants c_3, c_4 and a positive number $T_0 = c_3\mu(K)^{-(n+1)}$ such that for any $T \geq T_0$ there exist numbers $\alpha_1, \ldots, \alpha_{\mathbf{t}} \in \mathcal{A}_{n,p} \cap K$ such that*

$$\begin{gathered} H(\alpha_i) \leq T^{1/(n+1)} \ (1 \leq i \leq \mathbf{t}), \\ |\alpha_i - \alpha_j|_p \geq T^{-1} \ (1 \leq i < j \leq \mathbf{t}), \\ \mathbf{t} \geq c_4 T \mu(K). \end{gathered} \tag{1}$$

Note that from Theorem 1 it follows that the set $\mathcal{A}_{n,p}$ with the function $N(\alpha) = H^{n+1}(\alpha)$ form a regular system in K_0.

For $\bar{Q} \in \mathbb{R}^+$ define the set of polynomials

$$\mathcal{P}_n(\bar{Q}) = \{P \in \mathbb{Z}[x] : \deg P = n, \ \ H(P) \leq \bar{Q}\}. \tag{2}$$

To prove Theorem 1 it is convenient to introduce the following set. Let $Q \in \mathbb{N}$ and $\delta, d_n, c_5 \in \mathbb{R}^+$. We denote by $\bar{\mathcal{L}}_n = \bar{\mathcal{L}}_n(c_6 Q^{r_n}, \delta, K)$ the set of $w \in K$ for which the system of the inequalities

$$|P(w)|_p < c_5 Q^{-d_n}, \quad |P'(w)|_p \leq \delta, \tag{3}$$

has a solution in polynomials $P \in \mathcal{P}_n(c_6 Q^{r_n})$, where $c_6 \in \mathbb{R}^+$ and $0 \leq r_n \leq 1$. The proof of Theorem 1 is based on the following metric result which significantly broadens the scope of potential applications and is of independent interest.

Theorem 2. *For any real number l, where $0 < l < 1$, and for any cylinder K in $K_0 \subset \mathbb{Z}_p$ there exists a sufficiently large number $Q_0 = Q_0(K)$ such that for*

$$\mu(K) > c_7 Q_0^{-1}, \;\; d_n \geq n + r_n, \;\; \delta \leq 2^{-n-9} p^{-2} c_6^{-n-1} c_5^{-1} l^2 (s(n))^{-2}$$

and a sufficiently large constant c_7, which does not depend on Q_0, and for all $Q > Q_0$

$$\mu(\bar{\mathcal{L}}_n) < l\mu(K) \tag{4}$$

holds.

Remark 1. *The constant $s(n) \in \mathbb{N}$ is defined recursively in (65) and has the form*

$$s(n) = \begin{cases} 2 & \text{for } n = 1, \\ 14 & \text{for } n = 2, \\ 2n + 13 + \sum\limits_{k=3}^{n-1} s(k) + \sum\limits_{k=1}^{[n/2]} (4s(k) + 3s(n-k)) & \text{for } n \geq 3. \end{cases}$$

From above it follows that the cylinder K with $\mu(K) > c_7 Q^{-1}$ for sufficiently large c_7 and sufficiently large Q contains $\gg Q^{n+1}\mu(K)$ algebraic p-adic numbers of degree n and $H(\alpha) \leq Q$. Note that if $\mu(K) \leq \frac{1}{2}Q^{-1}$ then we have the following result which is a complement of Theorem 1 in some sense.

Theorem 3. *For any $Q \in \mathbb{N}$ there exist the cylinders K with $\mu(K) \leq \frac{1}{2}Q^{-1}$ which do not contain algebraic numbers $\alpha \in \mathbb{Q}_p$ of degree $\deg \alpha = n$, $n \geq 2$, and $H(\alpha) \leq Q$.*

2 Proof of Theorem 3

For the given Q choose $s \in \mathbb{N}$ satisfying the inequality $p^{-s} < \frac{1}{2}Q^{-1}$. Consider the cylinder $K = K(p^s, \frac{1}{2}Q^{-1})$. Let $\alpha \in K$ be an algebraic number of degree $\deg \alpha = n$, $n \geq 2$, and $H(\alpha) \leq Q$. It means that $\alpha \in \mathbb{Q}_p$, $\alpha \neq 0$, is a root of irreducible polynomial $P(x) = \sum_{i=0}^n a_i x^i$. If we assume that $a_0 = 0$ then from $P(\alpha) = 0$ it follows that $\alpha(\sum\limits_{i=1}^{n} a_i \alpha^{i-1}) = 0$. The last equation implies that α is a root of polynomial $P_1(x) = \sum\limits_{i=1}^{n} a_i x^{i-1}$ of $\deg P_1 \leq n-1$ which contradicts to the fact that $\deg \alpha = n$. Therefore, $a_0 \neq 0$ and from

$$a_0 = -\alpha \sum_{i=1}^{n} a_i \alpha^{i-1},$$

we obtain

$$Q^{-1} \leq |a_0|_p \leq |\alpha|_p \max_{1 \leq i \leq n} |a_i \alpha^{i-1}|_p \leq \frac{1}{2} Q^{-1},$$

which is a contradiction. This completes the proof of Theorem 3.

3 Proof of Theorem 2

By translation and taking the reciprocals (if necessary) each polynomial P can be transformed into a polynomial R satisfying

$$|a_n(R)|_p > c_8, \quad c_8 < 1, \tag{5}$$

and $H(R) \asymp H(P)$, see [15]. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the polynomial $P \in \mathbb{Z}[x]$ of degree n in $\mathbb{Q}_p^*$. Define the sets

$$S_P(\alpha_i) = \{w \in \mathbb{Q}_p : |w - \alpha_i|_p = \min_{1 \le j \le n} |w - \alpha_j|_p\}, \quad i = 1, \ldots, n.$$

We will consider the sets $S_P(\alpha_i)$ for a fixed i. For simplicity we assume that $i = 1$. Reorder the other roots of P so that

$$|\alpha_1 - \alpha_2|_p \le |\alpha_1 - \alpha_3|_p \le \ldots \le |\alpha_1 - \alpha_n|_p.$$

For the polynomial P define the real numbers ρ_j by

$$|\alpha_1 - \alpha_j|_p = H(P)^{-\rho_j}, \quad 2 \le j \le n, \quad \rho_2 \ge \rho_3 \ge \ldots \ge \rho_n.$$

Let $\epsilon > 0$ be sufficiently small, $d > 0$ be a large fixed number, $\epsilon_1 = \epsilon/d$ and $M = [\epsilon_1^{-1}]+1$. Also, define the integers l_j, $2 \le j \le n$, by the relations

$$\frac{l_j - 1}{M} \le \rho_j < \frac{l_j}{M}, \qquad l_2 \ge l_3 \ge \ldots \ge l_n \ge 0.$$

Finally, define the numbers q_i by $q_i = \frac{l_{i+1}+\ldots+l_n}{M}$, $(1 \le i \le n-1)$. All irreducible polynomials $P \in \mathcal{P}_n(c_6 Q^{r_n})$ satisfying (5) and corresponding to the same vector $\mathbf{l} = (l_2, \ldots, l_n)$ are grouped together into a class $\mathcal{P}_n(c_6 Q^{r_n}, \mathbf{l})$, and the number of such classes is finite and depends only on n and ϵ_1, i.e. is at most $C(n, \epsilon_1)$, see [15]. Also, we define the class $\mathcal{P}_n(\mathbf{l})$ to consist of all irreducible polynomials $P \in \mathbb{Z}[x]$ of degree n satisfying (5) and corresponding to a vector $\mathbf{l}$. In 3.2 we fix the vector $\mathbf{l}$ and will continue the proof for this fixed vector.

A number of lemmas for later use are now given.

Lemma 1. *[5] Let P be a polynomial without multiple zeros and let $w \in S_P(\alpha_1)$, then*

$$|w - \alpha_1|_p \le |P(w)|_p |P'(\alpha_1)|_p^{-1}, \tag{6}$$

$$|w - \alpha_1|_p \le \min_{2 \le j \le n} \left(|P(w)|_p |P'(\alpha_1)|_p^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_k|_p \right)^{\frac{1}{j}}. \tag{7}$$

Lemma 2. *[5] Let $w \in S_P(\alpha_1)$ and $|P'(w)|_p \ne 0$, then $|w - \alpha_1|_p \le |P(w)|_p |P'(w)|_p^{-1}$.*

Lemma 3. *[5] Let $P \in \mathcal{P}_n(\mathbf{l})$ satisfying (5). Then*

$$|P'(\alpha_1)|_p > c(n) H(P)^{-q_1} \quad \text{and} \quad |P^{(l)}(\alpha_1)|_p \le H(P)^{-q_l + (n-l)\epsilon_1}, \ 1 \le l \le n-1.$$

Lemma 4. *[16] Let $\theta > 0$ and $Q > Q_0(\theta)$. Further, let P_1 and P_2 be two integer polynomials of degree at most n with no common roots and $\max(H(P_1), H(P_2)) \leq Q$. Let $J \subset \mathbb{Q}_p$ be a cylinder with $\mu(J) = Q^{-\eta}$, $\eta > 0$. If there exists $\tau > 0$ such that for all $w \in J$*

$$|P_j(w)|_p < Q^{-\tau},$$

for $j = 1, 2$, then

$$\tau + 2\max(\tau - \eta, 0) < 2n + \theta. \tag{8}$$

Lemma 5. *Let $K \subset \mathbb{Q}_p$ be a cylinder and $B \subset K$ be a measurable set satisfying $\mu(B) \geq k^{-1}\mu(K) > 0$, $k \in \mathbb{N}$. Assume that for all $w \in B$ we have $|P(w)|_p < H(P)^{-a}$, where $a > 0$ and $\deg P \leq n$. Then for all $w \in K$ we have*

$$|P(w)|_p < (pk(n+1))^{n+1}H(P)^{-a}.$$

Proof. Let $\alpha = a_0 + a_1 p + \ldots + a_l p^l$ be center of the cylinder K. Then $K = \{w \in \mathbb{Q}_p : \ |w - \alpha|_p \leq p^{-(l+1)}\}$ with $\mu(K) = p^{-(l+1)}$ and $w = \alpha + a_{l+1}p^{l+1} + \ldots$. Choose s such that

$$k^{-1}p^s > n + 1. \tag{9}$$

Consider the cylinders $K(w_i)$:

$$K(w_i) = K(a_{l+1}, a_{l+2}, \ldots, a_{l+s}) = \{w \in \mathbb{Q}_p : |w - (\alpha + \sum_{i=1}^{s} a_{l+i}p^{l+i})|_p \leq p^{-(l+s+1)}\}.$$

It is clear that $\#K(w_i) = p^s$ and $K = \cup_{w_i} K(w_i)$, where $K(w_t) \cap K(w_m) = \emptyset$ for $t \neq m$. Let $B = K(w_{i_1}) \cup K(w_{i_2}) \cup \ldots \cup K(w_{i_r})$. Then

$$k^{-1}\mu(K) \leq \mu(B) = r\mu(K(w_i)) = rp^{-(l+s+1)} = rp^{-s}\mu(K)$$

and $r \geq k^{-1}p^s$. In the different cylinders $K(w_{i_u})$ and $K(w_{i_v})$, $u \neq v$, there exists a coordinate a_q of the vector $\mathbf{b}_{l,s} = (a_{l+1}, a_{l+2}, \ldots, a_{l+s})$ such that $a_q(K(w_{i_u})) \neq a_q(K(w_{i_v}))$, $l + 1 \leq q \leq l + s$. Therefore,

$$|w_u - w_v|_p \geq p^{-(l+1+s)},$$

where $w_u \in K(w_{i_u})$ and $w_v \in K(w_{i_v})$. Condition (9) allows us to choose at least $n + 1$ such points w_j.

Rewrite P as the interpolation polynomial in the Lagrange form

$$P(w) = \sum_{l=1}^{n+1} d_l \frac{(w - w_1)\ldots(w - w_{l-1})(w - w_{l+1})\ldots(w - w_{n+1})}{(w_l - w_1)\ldots(w_l - w_{l-1})(w_l - w_{l+1})\ldots(w_l - w_{n+1})}$$

where $d_l = P(w_l)$. Since $|w - w_i|_p \leq \mu(K)$ for all $w \in K$ and $|P(w_l)|_p < H(P)^{-a}$ then

$$|P(w)|_p < p^{s(n+1)}H(P)^{-a}.$$

Take $s = \log_p k(n+1) + 1 = \log_p pk(n+1)$ then $|P(w)|_p < (pk(n+1))^{n+1}H(P)^{-a}$ for all $w \in K$. □

Let $t \in (0, 1)$ be a sufficiently small number which we will specify later.

Lemma 6. *Denote by $L = L(c_9 Q^{r_1}, K)$ the set of $w \in K$, $\mu(K) > c_7 Q^{-1}$, for which the system of the inequalities*

$$|aw - b|_p < c_{10} Q^{-d_1}, \;\; \max(|a|, |b|) < c_9 Q^{r_1}, \;\; |a|_p \le c_{11} Q^{-v}, \tag{10}$$

has a solution in linear polynomials $aw - b \in \mathcal{P}_1(c_9 Q^{r_1})$, where the parameters $d_1 \ge 1$, $0 \le r_1 \le 1$, $v \ge 0$ and constants $c_i > 0$ satisfy one of the conditions:

i) $d_1 > 1 + r_1, \quad v \ge r_1 - 1,$
ii) $d_1 = 2, \quad r_1 = 1, \quad v = 0, \quad c_7 \ge 2^2 c_9 c_{10} t^{-1}, \quad 2^3 c_{10} c_9^2 c_{11} \le t,$
iii) $d_1 = 1 + r_1, \quad v > r_1 - 1, \quad c_7 \ge 2^2 c_9 c_{10} t^{-1}.$

Then $\mu(L) < 2t\mu(K)$ for Q sufficiently large.

Proof. Let $a = p^\beta a_1$ and $b = p^\beta b_1$, where $(a_1, p) = 1$, $p^{-\beta} \le c_{11} Q^{-v}$, $b_1 \in \mathbb{Z}$. Thus, we can rewrite (10) in the form

$$|a_1 w - b_1|_p < p^\beta c_{10} Q^{-d_1}, \;\; |a_1| < c_9 p^{-\beta} Q^{r_1}. \tag{11}$$

Now the measure of $w \in K$ for which the system (11) holds is estimated. For fixed a_1 and b_1 the first inequality in (11) holds for points $w \in K$ from the cylinder

$$|w - b_1/a_1|_p < p^\beta |a_1|_p^{-1} c_{10} Q^{-d_1} = p^\beta c_{10} Q^{-d_1}. \tag{12}$$

Then we need to sum the last estimate over all a_1 and b_1 such that $b_1/a_1 \in K$, where $|a_1| < c_9 p^{-\beta} Q^{r_1}$. For a fixed a_1 denote by $M_K(a_1)$ the number of such points b_1. For $M_K(a_1)$ the following formula holds:

$$M_K(a_1) \le \begin{cases} |a_1|\mu(K) + 1 \le 2|a_1|\mu(K) & if \quad |a_1| \ge \mu(K)^{-1}, \\ 1 & if \quad |a_1| < \mu(K)^{-1}. \end{cases} \tag{13}$$

Let $|a_1| \ge \mu(K)^{-1}$ and we use the first estimate in (13). Using $p^{-\beta} \le c_{11} Q^{-v}$, we obtain

$$\begin{gathered} \sum_{|a_1| < c_9 p^{-\beta} Q^{r_1}} \; \sum_{b_1 : b_1/a_1 \in K} p^\beta c_{10} Q^{-d_1} < 2^3 p^{-\beta} c_{10} c_9^2 Q^{2r_1 - d_1} \mu(K) \le \\ \le 2^3 c_{10} c_9^2 c_{11} Q^{2r_1 - d_1 - v} \mu(K) \le t\mu(K) \end{gathered} \tag{14}$$

for $2r_1 - d_1 - v < 0$ and $Q > Q_0$ or $2r_1 - d_1 - v = 0$ and $2^3 c_{10} c_9^2 c_{11} \le t$.

Let $|a_1| < \mu(K)^{-1}$ and we use the second estimate in (13). Summing over a_1 and b_1 we get

$$\begin{gathered} \sum_{|a_1| < c_9 p^{-\beta} Q^{r_1}} \; \sum_{b_1 : b_1/a_1 \in K} p^\beta c_{10} Q^{-d_1} < 4 c_{10} c_9 Q^{r_1 - d_1} < \\ < 4 c_{10} c_9 c_7^{-1} Q^{1 + r_1 - d_1} \mu(K) \le t\mu(K) \end{gathered} \tag{15}$$

for $1 + r_1 - d_1 < 0$ and $Q > Q_0$ or $1 + r_1 - d_1 = 0$ and $c_7 \ge 2^2 t^{-1} c_9 c_{10}$. □

Now consider the special case when $r_n = 0$. Denote by $L_0 = L_0(c_6, K)$ the set of $w \in K$, $\mu(K) > c_7 Q^{-1}$, for which the system

$$|P(w)|_p < c_5 Q^{-d_n}, \quad d_n \ge n \tag{16}$$

has a solution in $P \in \mathcal{P}_n(c_6)$. Let $\sigma'(P)$ denote the set of w of (16) for a fixed polynomial $P \in \mathcal{P}_n(c_6)$. Let $w \in \sigma'(P) \cap S_P(\alpha_1)$ for some $P \in \mathcal{P}_n(c_6)$. Then by (16) and Lemma 1, we have

$$|w - \alpha_1|_p < (c_5 c_8^{-1} Q^{-d_n})^{1/n}.$$

Summing the last estimate over all polynomials $P \in \mathcal{P}_n(c_6)$, we get

$$\mu(L_0) < n c_5^{1/n} (2c_6 + 1)^{n+1} c_8^{-1/n} Q^{-d_n/n} \leq t\mu(K)$$

for $c_7 \geq c_5^{1/n}(2c_6 + 1)^{n+1} c_8^{-1/n} t^{-1} n$. From now on assume that $r_n > 0$.

Note that we will prove Theorem 2 by strong induction with the following *induction hypothesis:* assume that for $1 \leq m \leq n-1$ the following

$$\mu\left(w \in K : \exists P \in \mathcal{P}_m(m_2 Q^{r_m}) s.t. \begin{array}{l} |P(w)|_p < m_1 Q^{-d_m}, \\ |P'(w)|_p \leq \delta, \\ d_m \geq m + r_m, \\ \delta \leq 2^{-m-9} p^{-2} m_1^{-1} m_2^{-(m+1)} t^2 \end{array} \right) < s(m) t \mu(K)$$

holds for sufficiently large c_7 and sufficiently large Q, where $\mu(K) > c_7 Q^{-1}$ and $s(m) \in \mathbb{N}$ is constant depending on the degree m of a polynomial. The *base case* for $m = 1$ with $s(1) = 2$ follows from Lemma 6.

3.1 Case of large derivative

Define the subset $\tilde{\mathcal{L}}_n$ of the set $\bar{\mathcal{L}}_n$ containing $w \in K$ for which there exists polynomial $P \in \mathcal{P}_n(c_6 Q^{r_n})$ such that the system

$$|P(w)|_p < c_5 Q^{-d_n}, \quad p c_5^{1/2} Q^{-d_n/2} < |P'(w)|_p \leq \delta \tag{17}$$

holds.

Denote by $\sigma_0(P)$ the set of solutions w of the system (17) for a fixed polynomial $P \in \mathcal{P}_n(c_6 Q^{r_n})$. Then we have $\tilde{\mathcal{L}}_n = \bigcup_{P \in \mathcal{P}_n(c_6 Q^{r_n})} \sigma_0(P)$. Let $P \in \mathcal{P}_n(c_6 Q^{r_n})$ and $w \in \sigma_0(P) \cap S_P(\alpha_1)$ where $P(\alpha_1) = 0$. By the Taylor's formula

$$P'(w) = \sum_{i=1}^{n} ((i-1)!)^{-1} P^{(i)}(\alpha_1)(w - \alpha_1)^{i-1}.$$

Using $|w - \alpha_1|_p < c_5 Q^{-d_n} |P'(w)|_p^{-1}$ from Lemma 1 and estimating each term gives

$$|P'(\alpha_1)|_p = |P'(w)|_p.$$

Therefore, the set $\sigma_0(P) \cap S_P(\alpha_1)$ is contained in $\sigma(P)$ which is defined by

$$|w - \alpha_1|_p < c_5 Q^{-d_n} |P'(\alpha_1)|_p^{-1}. \tag{18}$$

Further to obtain the measure of $\tilde{\mathcal{L}}_n$ it is necessary to consider several cases which depend on the value of $|P'(\alpha_1)|_p$ in the range $(p c_5^{1/2} Q^{-d_n/2}, \delta]$.

3.1.1 Case A: $2^{(n+1)/2}pc_6^{(n-2)/2}c_5^{1/2}t^{-1/2}Q^{-(2+r_n)/2} < |P'(\alpha_1)|_p \leq \delta$

Define the subset $\mathcal{L}_{n1}$ of the set $\tilde{\mathcal{L}}_n$ for which there exists at least one polynomial $P \in \mathcal{P}_n(c_6 Q^{r_n})$ satisfying (17) and the inequality

$$Q^{-r_n/2} < |P'(\alpha_1)|_p \leq \delta, \tag{19}$$

where α_1 is the closest root to w of P.

Proposition 1. *For $\delta \leq 2^{-n-5}c_6^{-n-1}c_5^{-1}t^2$ and sufficiently large constant c_7 and sufficiently large Q we have*

$$\mu(\mathcal{L}_{n1}) < 3t\mu(K).$$

Proof. For a polynomial $P \in \mathcal{P}_n(c_6 Q^{r_n})$ define the cylinder

$$\sigma_{1,1}(P) := \{w \in S_P(\alpha_1) \cap K : \ |w - \alpha_1|_p < c_{12}Q^{-(1+r_n)}|P'(\alpha_1)|_p^{-1}\}. \tag{20}$$

From (18) and (20) we get

$$\mu(\sigma(P)) < c_5 c_{12}^{-1} Q^{-d_n+1+r_n}\mu(\sigma_{1,1}(P)). \tag{21}$$

Note that from (19) it follows that $\mu(\sigma_{1,1}(P)) < c_{12}Q^{-1-r_n/2}$ and $\mu(\sigma_{1,1}(P)) < \mu(K)$ for $c_7 \geq c_{12}$.

Decompose the polynomial P into Taylor series on the cylinder $\sigma_{1,1}(P)$ so that

$$P(w) = \sum_{i=1}^{n}(i!)^{-1}P^{(i)}(\alpha_1)(w-\alpha_1)^i.$$

Using (19) and (20), estimate each term of the decomposition to obtain

$$|P(w)|_p < c_{12}Q^{-1-r_n} \quad \text{for } Q > Q_0. \tag{22}$$

Let $w \in \sigma_{1,1}(P)$. By Taylor's formula,

$$|P'(w)|_p \leq \delta \quad \text{for } Q > Q_0. \tag{23}$$

Fix the vector $\mathbf{b}_1 = (a_n, \ldots, a_2)$ which consists of the coefficients of the polynomial $P(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{P}_n(c_6 Q^{r_n})$. Let the subclass of polynomials $P \in \mathcal{P}_n(c_6 Q^{r_n})$ with the same vector $\mathbf{b}_1$ be denoted by $\mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_1)$. The cylinders $\sigma_{1,1}(P)$ divide into two classes using Sprindzuk's method of essential and inessential domains [15]. The cylinders $\sigma_{1,1}(P)$ are called *inessential* if there is a polynomial $\bar{P} \in \mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_1)$ (with $P \neq \bar{P}$), such that

$$\mu(\sigma_{1,1}(P) \cap \sigma_{1,1}(\bar{P})) \geq 1/2\mu(\sigma_{1,1}(P)), \tag{24}$$

and *essential* otherwise. According to this classification, we have $\mathcal{L}_{n1} \subseteq \mathcal{V}_{ess} \cup \mathcal{V}_{iness}$.

First, the essential cylinders $\sigma_{1,1}(P)$ are investigated. By definition

$$\sum_{P \in \mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_1)} \mu(\sigma_{1,1}(P)) \leq \mu(K).$$

Using the last estimate, (21) and the fact that the number of different vectors $\mathbf{b}_1$ does not exceed $(2c_6Q^{r_n}+1)^{n-1}$, it follows that

$$\mu(\mathcal{V}_{ess}) = \sum_{\mathbf{b}_1} \sum_{\substack{P\in\mathcal{P}_n(c_6Q^{r_n},\mathbf{b}_1)\\ \sigma_{1,1}(P)\text{ essential}}} \mu(\sigma(P)) < 2^n c_6^{n-1} c_5 c_{12}^{-1} Q^{-d_n+1+nr_n}\mu(K) \le t\mu(K) \qquad (25)$$

for $c_{12} \ge 2^n c_6^{n-1} c_5 t^{-1}$ and $Q > Q_0$.

Second, we consider the inessential cylinders $\sigma_{1,1}(P)$. Let $\sigma_{1,1}(P,\bar{P}) = \sigma_{1,1}(P) \cap \sigma_{1,1}(\bar{P})$, where $P, \bar{P} \in \mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_1)$ and $P \ne \bar{P}$. Then on the set $\sigma_{1,1}(P,\bar{P})$ with the measure at least $1/2\mu(\sigma_{1,1}(P))$ for the polynomials P and $\bar{P}$ the inequality (22) holds. Now consider the new polynomial $R(w) = P(w) - \bar{P}(w)$ which is a linear polynomial since the polynomials P and $\bar{P}$ have the same coefficients $a_n, a_{n-1}, \dots, a_2$. Thus, by Lemma 5, (22) and (23) for $w \in \sigma_{1,1}(P)$ we have

$$|R(w)|_p = |aw-b|_p < 2^4p^2c_{12}Q^{-1-r_n}, \;\; \max(|a|,|b|) < 2c_6Q^{r_n}, \;\; |a|_p \le \delta. \qquad (26)$$

Denote by $L_1(2c_6Q^{r_n}, K)$ the set of $w \in K$ for which the system (26) has a solution in polynomials $P \in \mathcal{P}_1(2c_6Q^{r_n})$. By Lemma 6(ii, iii), we have $\mu(L_1(2c_6Q^{r_n}, K)) < 2t\mu(K)$ for $c_7 \ge 2^7p^2c_6c_{12}t^{-1}$ and $\delta \le 2^{-9}p^{-2}c_6^{-2}c_{12}^{-1}t$. Obviously $\mathcal{V}_{iness} \subseteq L_1(2c_6Q^{r_n}, K)$.

Choose $c_{12} = 2^n c_5 t^{-1} c_6^{n-1}$. Therefore, for the measure of the set $\mathcal{L}_{n1}(c_6Q^{r_n})$ the bounds, obtained for both essential and inessential cylinders, can be rewritten as

$$\mu(\mathcal{L}_{n1}) < 3t\mu(K) \qquad (27)$$

for $\delta \le 2^{-n-9}p^{-2}c_5t^{-1}c_6^{-n-1}t^2$ and $c_7 \ge \max\{2^{n+7}p^2c_5c_6^nt^{-2}, 2^nc_5t^{-1}c_6^{n-1}\}$. This completes the proof of Proposition 1. □

For some $c_{13} > 0$ define the subset $\mathcal{L}_{n2}$ of the set $\tilde{\mathcal{L}}_n$, containing the $w \in K$, for which there exists at least one polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$ satisfying (17) and the inequality

$$c_{13}Q^{-r_n} < |P'(\alpha_1)|_p \le Q^{-r_n/2},$$

where α_1 is the closest root to w of P.

Proposition 2. *For $c_{13} = 2^{n/2+1}pc_5^{1/2}c_6^{(n-1)/2}t^{-1/2}$ and sufficiently large constant c_7 and sufficiently large Q we have $\mu(\mathcal{L}_{n2}) < 3t\mu(K)$.*

Proof. The proof of the Proposition 2 is closely related to the proof of Proposition 1. As before, for $P \in \mathcal{P}_n(c_6Q^{r_n})$ and some positive constant c_{14} (which will be specified later) we consider the cylinder $\sigma(P)$ and define the cylinder

$$\sigma_{1,2}(P) := \{w \in S_P(\alpha_1) \cap K : \; |w-\alpha_1|_p < c_{14}Q^{-1-r_n}|P'(\alpha_1)|_p^{-1}\}. \qquad (28)$$

It is clear that

$$\mu(\sigma(P)) < c_{14}^{-1}Q^{-d_n+1+r_n}\mu(\sigma_{1,2}(P)). \qquad (29)$$

The definition of $\mathcal{L}_{n2}$ gives us that $\mu(\sigma_{1,2}(P)) < \mu(K)$ for $c_7 \ge c_{13}^{-1}c_{14}$. Develop P and P' as a Taylor series on $\sigma_{1,2}(P)$ to obtain

$$|P(w)|_p < c_{14}Q^{-1-r_n}, \quad |P'(w)|_p = |P'(\alpha_1)|_p \qquad (30)$$

for $c_{14} < p^{-2}c_{13}^2$. Further consider the essential and inessential cylinders $\sigma_{1,2}(P)$. In the case of the essential cylinders we have

$$\sum_{P\in\mathcal{P}_n(c_6Q^{r_n},\mathbf{b}_1)} \mu(\sigma_{1,2}(P)) \le \mu(K),$$

$$\sum_{\mathbf{b}_1}\sum_{P\in\mathcal{P}_n(c_6Q^{r_n},\mathbf{b}_1)} \mu(\sigma(P)) < 2^n c_6^{n-1}c_5c_{14}^{-1}Q^{-d_n+1+nr_n}\mu(K) \le t\mu(K) \tag{31}$$

for $c_{14} \ge 2^n c_6^{n-1}c_5t^{-1}$ and $Q > Q_0$.

It follows from (30) that in the case of the inessential cylinders for the polynomial $T(w) = P(w) - \bar{P}(w) = kw - d$, where $P, \bar{P} \in \mathcal{P}_n(c_6Q^{r_n})$, and $P \neq \bar{P}$. By (30) and Lemma 5, for $w \in \sigma_{1,2}(P)$ we have

$$|kw-d|_p < 2^4p^2c_{14}Q^{-1-r_n}, \quad \max(|k|,|d|) < 2c_6Q^{r_n}, \quad |k|_p \le Q^{-r_n/2}. \tag{32}$$

Denote by $L_2(2c_6Q^{r_n}, K)$ the set of $w \in K$ for which the system (32) has a solution in polynomials $P \in \mathcal{P}_1(2c_6Q^{r_n})$. By Lemma 6(iii), we obtain that $\mu(L_2(2c_6Q^{r_n}, K)) < 2t\mu(K)$ for $c_7 \ge 2^7p^2c_6c_{14}t^{-1}$.

Choose $c_{14} = 2^n c_6^{n-1}c_5t^{-1}$ and $c_{13} = 2^{n/2+1}pc_6^{(n-1)/2}c_5^{1/2}t^{-1/2}$. The upshot is that

$$\mu(\mathcal{L}_{n2}) < 3t\mu(K) \tag{33}$$

for $c_7 \ge \max(2^{n/2-1}p^{-1}c_6^{(n-1)/2}c_5^{1/2}t^{-1/2}, 2^{n+7}p^2c_6^nc_5t^{-2})$. This completes the proof of Proposition 2. □

In the case if $c_6^{(n+1)/2}c_5^{1/2} > 2^{-(n+10)/2}p^{-1}t^{1/2}$ we need to consider the following set. Denote by $\mathcal{L}_{n3} \subset \tilde{\mathcal{L}}_n$ the set of $w \in K$, for which there exists at least one polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$ satisfying (17) and the inequality

$$2^{-4}c_6^{-1}Q^{-r_n} < |P'(\alpha_1)|_p \le 2^{n/2+1}pc_6^{(n-1)/2}c_5^{1/2}t^{-1/2}Q^{-r_n},$$

where α_1 is the closest root to w of P.

Proposition 3. *For sufficiently large constant c_7 and sufficiently large Q we have $\mu(\mathcal{L}_{n3}) < 3t\mu(K)$.*

Proof. For $P \in \mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_1)$ and some $c_{15} > 0$ define the cylinder

$$\sigma_{1,3}(P) := \{w \in S_P(\alpha_1) \cap K : \ |w-\alpha_1|_p < c_{15}Q^{-(1+r_n)}|P'(\alpha_1)|_p^{-1}\}.$$

The definition of $\mathcal{L}_{n3}$ gives us that $\mu(\sigma_{1,3}(P)) < \mu(K)$ for $c_7 \ge 2^4c_6c_{15}$. Develop P and P' as a Taylor series on $\sigma_{1,3}(P)$ to obtain

$$|P(w)|_p \le c_{16}Q^{-1-r_n}, \quad |P'(w)|_p \le c_{17}Q^{-r_n}$$

for $c_{16} = \max(c_{15}, 2^8p^2c_6^2c_{15}^2)$ and $c_{17} = \max(2^{n/2+1}pc_6^{(n-1)/2}c_5^{1/2}t^{-1/2}, 2^4pc_6c_{15})$.

Then consider the essential and inessential cylinders $\sigma_{1,3}(P)$ for $P \in \mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_1)$. In the case of the essential cylinders we obtain that the measure does not exceed $t\mu(K)$

for $c_{15} \geq 2^n c_5 c_6^{n-1} t^{-1}$. In the case of the inessential cylinders we need to find the measure of $w \in K$ for which there exists at least one polynomial $P \in \mathcal{P}_1(2c_6 Q^{r_n})$ satisfying

$$|aw - b|_p < 2^4 p^2 c_{16} Q^{-1-r_n}, \quad |a|_p < c_{17} Q^{-r_n} \tag{34}$$

for any $w \in \sigma_{1,3}(P)$. By Lemma 6(iii), the measure in the case of inessential domains is at most $2t\mu(K)$ for $c_7 \geq 2^7 p^2 c_6 c_{16} t^{-1}$. Choose $c_{15} = 2^n c_6^{n-1} c_5 t^{-1}$. Then we get $c_7 \geq \max(2^{n+4} c_6^n c_5 t^{-1}, 2^7 p^2 c_6 c_{16} t^{-1})$. $\square$

For some constant $c_{18} > 0$ we denote by $\mathcal{L}_{n4} \subset \tilde{\mathcal{L}}_n$ the set of $w \in K$, for which there exists at least one polynomial $P \in \mathcal{P}_n(c_6 Q^{r_n})$ satisfying (17) and the inequality

$$c_{18} Q^{-(2+r_n)/2} < |P'(\alpha_1)|_p \leq 2^{-4} c_6^{-1} Q^{-r_n},$$

where α_1 is the closest root to w of P.

Proposition 4. *For $c_{18} = 2^{(n+1)/2} p c_5^{1/2} c_6^{(n-2)/2} t^{-1/2}$ and sufficiently large constant c_7 and sufficiently large Q we have $\mu(\mathcal{L}_{n4}) < 3t\mu(K)$.*

Proof. For $P \in \mathcal{P}_n(c_6 Q^{r_n})$ and some $c_{19} > 1$ define the cylinder

$$\sigma_2(P) := \{w \in S_P(\alpha_1) \cap K : \ |w - \alpha_1|_p < c_{19} Q^{-(2+r_n)} |P'(\alpha_1)|_p^{-1}\}.$$

Clearly, that

$$\mu(\sigma(P)) < c_5 c_{19}^{-1} Q^{-d_n+2+r_n} \mu(\sigma_2(P)). \tag{35}$$

The definition of $\mathcal{L}_{n4}$ gives us that $\mu(\sigma_2(P)) < \mu(K)$ for $c_7 \geq c_{18}^{-1} c_{19}$.

Fix $\mathbf{b}_2 = (a_n, \ldots, a_3)$. Let the subclass of polynomials $P \in \mathcal{P}_n(c_6 Q^{r_n})$ with the same vector $\mathbf{b}_2$ be denoted by $\mathcal{P}_n(c_6 Q^{r_n}, \mathbf{b}_2)$. Consider again essential and inessential domains $\sigma_2(P)$ for $P \in \mathcal{P}_n(c_6 Q^{r_n}, \mathbf{b}_2)$.

By the definition of the essential domains, it follows that

$$\sum_{P \in \mathcal{P}_n(c_6 Q^{r_n}, \mathbf{b}_2)} \mu(\sigma_2(P)) \leq \mu(K).$$

Since the number of $\mathbf{b}_2$ does not exceed $(2c_6 Q^{r_n} + 1)^{n-2}$ then, summing over all $\mathbf{b}_2$ and using (35) and $d_n \geq n + r_n$, we have

$$\sum_{\mathbf{b}_2} \sum_{P \in \mathcal{P}_n(c_6 Q^{r_n}, \mathbf{b}_2)} \mu(\sigma(P)) < 2^{n-1} c_6^{n-2} c_5 c_{19}^{-1} Q^{r_n(n-1) - d_n + 2} \mu(K) \leq$$
$$\leq 2^{n-1} c_6^{n-2} c_5 c_{19}^{-1} Q^{(r_n - 1)(n-2)} \mu(K) \leq t\mu(K)$$

for $c_{19} \geq 2^{n-1} c_6^{n-2} c_5 t^{-1}$, $n \geq 2$ and $Q > Q_0$.

Now consider the inessential domains. By the Taylor expansion of $P_i(w)$ and $P_i'(w)$ on $\sigma_2(P_{i_1}, P_{i_2}) = \sigma_2(P_{i_1}) \cap \sigma_2(P_{i_2})$, $P_{i_1}, P_{i_2} \in \mathcal{P}_n(c_6 Q^{r_n}, \mathbf{b}_2)$ $P_{i_1} \neq P_{i_2}$, find the upper bound of $|P_i(w)|_p$ and $|P_i'(w)|_p$, so that

$$|P_i(w)|_p < c_{19} Q^{-2-r_n}, \quad |P_i'(w)|_p = |P'(\alpha_1)|_p \ \text{ for } \ c_{18} > p c_{19}^{1/2}. \tag{36}$$

Since the leading coefficients of P_{i_1} and P_{i_2} are equal then $W(w) = P_{i_1}(w) - P_{i_2}(w) = f_2w^2 + f_1w + f_0$ and, by (36),

$$|W(w)|_p < c_{19}Q^{-2-r_n}, \quad |W'(w)|_p < |P'(\alpha_1)|_p, \quad |f_i| \le 2c_6Q^{r_n}, \quad 0 \le i \le 2. \tag{37}$$

Then we need to consider the discriminant $D(W)$ of W and distinguish two cases: $D(W) \ne 0$ and $D(W) = 0$. It is easy to verify that the representation of $D(P)$ for $P \in \mathcal{P}_n(2c_6Q^{r_n})$ as a determinant leads to the upper bound

$$|D(P)| \le 2n^{2n-1}(2n-2)!(2c_6Q^{r_n})^{2n-2}.$$

Case 1: $D(W) \ne 0$. Let $\beta_1, \beta_2 \in \mathbb{Q}_p^*$ denote the roots of $W(w)$. Since the discriminant $D(W)$ of W satisfies

$$|D(W)|_p = |W'(\beta_1)|_p^2 < |P'(\alpha_1)|_p^2 \le 2^{-8}c_6^{-2}Q^{-2r_n},$$
$$|D(W)|_p \ge |D(W)|^{-1} \ge 2^{-7}c_6^{-2}Q^{-2r_n}$$

then we have a contradiction.

Case 2: $D(W) = 0$. This implies that the polynomial W has a multiple root and has a form

$$W(w) = W_1^2(w) = (l_1w - l_0)^2,$$

where by Gelfond's Lemma [11] we have $\max(|l_1|, |l_0|) \le 2^{(n+1)/2}c_6^{1/2}Q^{r_n/2}$. By (37) and Lemma 5, we have

$$|l_1w - l_0|_p < 2^4p^2c_{19}^{1/2}Q^{-(2+r_n)/2} \tag{38}$$

for any $w \in \sigma_2(P_{i_1})$. Denote by $L_3(2^{(n+1)/2}c_6^{1/2}Q^{r_n/2}, K)$ the set of $w \in K$ for which the inequality (38) has a solution in polynomials $P \in \mathcal{P}_1(2^{(n+1)/2}c_6^{1/2}Q^{r_n/2})$. By Lemma 6(iii), we have $\mu(L_3(2^{(n+1)/2}c_6^{1/2}Q^{r_n/2}, K))) < 2t\mu(K)$ for $c_7 \ge 2^{(n+13)/2}p^2c_6^{1/2}c_{19}^{1/2}t^{-1}$.

Choose $c_{19} = 2^{n-1}c_6^{n-2}c_5t^{-1}$ and $c_{18} = 2^{(n+1)/2}pc_6^{(n-2)/2}c_5^{1/2}t^{-1/2}$. Then sum the estimates for the measure of the essential and inessential cases. For

$$c_7 \ge \max(2^{(n-3)/2}p^{-1}c_6^{(n-2)/2}c_5^{1/2}t^{-1/2}, 2^{n+6}p^2c_6^{(n-1)/2}c_5^{1/2}t^{-3/2})$$

this concludes the proof of Proposition 4. □

Remark 2. *For $n = 2$ after Proposition 4 we need to use the following argument to finish the proof of theorem. It is easy to show that we left with the case when $|P'(\alpha_1)|_p \le c_{18}Q^{-(2+r_2)/2}$. Similar as in Proposition 4 we obtain that $D(P) = 0$. Therefore, we have $P(w) = (aw+b)^2$ which implies that $|aw+b|_p < c_5^{1/2}Q^{-d_2/2}$ and $\max(|a|, |b|) < 2c_6^{1/2}Q^{r_2/2}$. By Lemma 6(i,iii) we have that the measure of $w \in K$, for which there exists at least one linear polynomial $P \in \mathcal{P}_1(2c_6^{1/2}Q^{r_2/2})$ satisfying the last inequalities, does not exceed $2t\mu(K)$ for $d_2 > r_2 + 2$ or $d_2 = r_2 + 2$ and $c_7 \ge 2^3c_6^{1/2}c_5^{1/2}t^{-1}$.*

Further, we assume that $n \ge 3$.

3.1.2 Case B: $c_{20}Q^{-(n-1+r_n)/2} < |P'(\alpha_1)|_p \le c_{18}Q^{-(2+r_n)/2}$

Here c_{20} is a sufficiently small constant which will be specified in Subsection 3.3.

Let $3 \le k \le n-1$. Consider the following ranges for the value of first derivative:

$$v_k Q^{-(k+r_n)/2} \quad < \quad |P'(\alpha_1)|_p \quad \le v'_k Q^{-(k-1+r_n)/2}, \tag{39}$$

where $v_3 = v'_{n-1} = 1$, $v'_3 = c_{18}$, $v_{n-1} = c_{20}$ and $v_k = v'_k = 1$ for $4 \le k \le n-2$.

For $3 \le k \le n-1$ denote by $\mathcal{L}_{n,k} \subset \tilde{\mathcal{L}}_n$ the set of $w \in K$, for which there exists at least one polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$ satisfying (17) and (39).

Proposition 5. *For sufficiently large constant c_7 and sufficiently large Q we have $\mu(\mathcal{L}_{n,k}) < (s(k)+1)t\mu(K)$.*

Proof. For a polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$ define the cylinder

$$\sigma_k(P) := \{w \in S_P(\alpha_1) \cap K : \ |w-\alpha_1|_p < c_{21}Q^{-(k+r_n)}|P'(\alpha_1)|_p^{-1}\},\ 3 \le k \le n.$$

For $3 \le k \le n-1$ fix the vector $\mathbf{b}_k = (a_n, \ldots, a_{k+1})$. Let the subclass of polynomials $P \in \mathcal{P}_n(c_6Q^{r_n})$ with the same vector $\mathbf{b}_k$ be denoted by $\mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_k)$. The cylinders $\sigma_k(P)$ divide into two classes of essential and inessential domains. For $Q > Q_0$ we will use the estimate $\#\{\mathbf{b}_k\} < 2^{n-k+1}c_6^{n-k}Q^{r_n(n-k)}$.

First, the essential cylinders $\sigma_k(P)$ are investigated. By definition

$$\sum_{P \in \mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_k)} \mu(\sigma_k(P)) \le \mu(K).$$

Using the last estimate, (18) and the fact that the number of different vectors $\mathbf{b}_k$ does not exceed $2^{n-k+1}c_6^{n-k}Q^{r_n(n-k)}$, it follows that

$$\begin{aligned}\sum_{\mathbf{b}_k} \sum_{P \in \mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_k)} \mu(\sigma(P)) &< 2^{n+1-k}c_6^{n-k}c_5c_{21}^{-1}Q^{r_n(n-k+1)-d_n+k}\mu(K) \le \\ &\le 2^{n+1-k}c_6^{n-k}c_5c_{21}^{-1}Q^{(n-k)(r_n-1)}\mu(K) \le t\mu(K)\end{aligned} \tag{40}$$

for $c_{21} \ge 2^{n+1-k}c_6^{n-k}c_5t^{-1}$.

Second, we consider the inessential cylinders $\sigma_k(P)$. Let $\sigma_k(P, \bar{P}) = \sigma_k(P) \cap \sigma_k(\bar{P})$, where $P, \bar{P} \in \mathcal{P}_n(c_6Q^{r_n}, \mathbf{b}_k)$ and $P \ne \bar{P}$. Then on the set $\sigma_k(P, \bar{P})$ with the measure at least $1/2\mu(\sigma_k(P))$ for the polynomials P and $\bar{P}$ the following system holds:

$$|P(w)|_p < c_{22}Q^{-k-r_n}, \qquad |P'(w)|_p \le v'_kQ^{-(k-1+r_n)/2}, \tag{41}$$

where $c_{22} = max\{c_{21}, p^2c_{21}^2v_k^{-2}\}$. According to Lemma 5 and (41), for the new polynomials $R(w) = P(w) - \bar{P}(w)$ of $\deg R \le k$ with $H(R) \le 2c_6Q^{r_n}$ on $\sigma_k(P)$ we have

$$|R(w)|_p < (2p(k+1))^{k+1}c_{22}Q^{-k-r_n}, \qquad |R'(w)|_p \le (2pk)^k v'_kQ^{-(k-1+r_n)/2}. \tag{42}$$

By applying inductive hypothesis to polynomials R and using (40), we obtain $\mu(\mathcal{L}_{n,k}) < (s(k)+1)t\mu(K)$ for $3 \le k \le n-1$, sufficiently large c_7 and sufficiently large Q. □

It now follows via Proposition 5, that $\mu(\bigcup_{k=3}^{n-1} \mu(\mathcal{L}_{n,k})) < (\sum_{k=3}^{n-1} s(k) + n - 3)t\mu(K)$ for $Q > Q_0$ and sufficiently large c_7.

3.1.3 Case C: $pc_5^{1/2}Q^{-d_n/2} < |P'(\alpha_1)|_p \le c_{20}Q^{-(n-1+r_n)/2}$ and irreducible polynomials

Consider the set $\mathcal{L}_{n,n}$ which is the set of $w \in K$, for which there exists at least one irreducible polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$ satisfying

$$|P(w)|_p < c_5Q^{-d_n}, \qquad pc_5^{1/2}Q^{-d_n/2} < |P'(\alpha_1)|_p \le c_{20}Q^{-(n-1+r_n)/2}. \tag{43}$$

Proposition 6. *For sufficiently large Q we have $\mu(\mathcal{L}_{n,n}) < 2t\mu(K)$.*

Proof. Divide the cylinder K into smaller cylinders J_i with $\mu(J_i) = Q^{-u}$ where $u > 1$. We say the polynomial P belongs to the cylinder J_i if there exists $w \in J_i$ such that (3) and (43) hold. If there is at most one irreducible polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$ that belongs to every J_i then by Lemma 1 the measure of those w, that satisfy (43), does not exceed

$$np^{-1}c_5^{1/2}Q^{-d_n/2+u}\mu(K) < t\mu(K) \tag{44}$$

for $u < d_n/2$ and sufficiently large Q.

If at least two irreducible polynomials $P_i \in \mathcal{P}_n(c_6Q^{r_n})$ of the form $P_i(w) = k_iP(w)$ for the same irreducible polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$, $k_i \in \mathbb{Z}$, belong to the cylinder J_i then the measure in this case coincides with the measure in (44).

The assumption that at least two irreducible polynomials without common roots belong to the cylinder J_i will lead to a contradiction. To show this, suppose that P_1 and P_2 belong to J_i. Develop P_1 as a Taylor series in the neighbourhood J_i of α_1 to obtain

$$|P(w)|_p \le \max\{c_{20}Q^{-(n-1+r_n)/2-u}, p^2Q^{-2u}\} = c_{20}Q^{-(n-1+r_n)/2-u}, \quad w \in J_i,$$

for $u > (n-1+r_n)/2$. Obviously, the same estimate holds for P_2 on J_i.

Applying Lemma 4 to polynomials P_1 and P_2 with $\tau = ((n-1+r_n)/2 + u - \epsilon_1')/r_n$ and $\eta = (u+\epsilon_2')/r_n$, where $\epsilon_i' > 0$ is sufficiently small, leads to a contradiction in (8) for $u > (n-1+r_n)/2+2\theta$ and $\epsilon_1'+\epsilon_2' \le \theta$. Choose u, satisfying $(n-1+r_n)/2+2\theta < u < d_n/2$. □

3.2 Case of small derivative and irreducible polynomials

Define the subset $\check{\mathcal{L}}_n$ of the set $\bar{\mathcal{L}}_n$ containing $w \in K$ for which there exists irreducible polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$ such that

$$|P(w)|_p < c_5Q^{-d_n}, \quad |P'(w)|_p \le pc_5^{1/2}Q^{-d_n/2}. \tag{45}$$

Proposition 7. *For sufficiently large constant c_7 and sufficiently large Q we have $\mu(\check{\mathcal{L}}_n) < 3t\mu(K)$.*

Proof. Define by $\sigma_*(P)$ the set of solutions of the system (45) for a fixed polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$. Let $w \in \sigma_*(P) \cap S_P(\alpha_1)$. First, it is shown that the value of the derivative of P at α_1, $P(\alpha_1) = 0$, satisfies

$$|P'(\alpha_1)|_p \le pc_5^{1/2}Q^{-d_n/2}. \tag{46}$$

To show this, assume the opposite of (46). Then develop P' as a Taylor series in the neighborhood of α_1 and use the estimate $|w-\alpha_1|_p < c_5^{1/2}p^{-1}Q^{-d_n/2}$ from Lemma 1. Since

$$\max\{\max_{2\le j\le n}\{|((j-1)!)^{-1}P^{(j)}(\alpha_1)|_p|w-\alpha_1|_p^{j-1}\}, |P'(w)|_p\} \le c_5^{1/2}p^{-1}Q^{-d_n/2}$$

for $Q > Q_0$, it follows that $|P'(\alpha_1)|_p \le c_5^{1/2}p^{-1}Q^{-d_n/2}$ which contradicts to the condition that $|P'(\alpha_1)|_p > pc_5^{1/2}Q^{-d_n/2}$.

Note that the set $\check{\mathcal{L}}_n$ can be written as

$$\check{\mathcal{L}}_n = \begin{cases} L_{\le} & \text{if} \quad d_n > n + n(n+1)r_n, \\ L_{\le} \cup L_{>} & \text{if} \quad d_n \le n + n(n+1)r_n, \end{cases}$$

where $L_{\le} = \bigcup_{P\in\mathcal{P}_n\left(Q^{\frac{d_n-n}{n(n+1)}}\right)} \sigma_*(P)$ and $L_{>} = \bigcup_{P\in\mathcal{P}_n(c_6Q^{r_n})\setminus\mathcal{P}_n\left(Q^{\frac{d_n-n}{n(n+1)}}\right)} \sigma_*(P)$.

Next, we are going to establish the following two separate cases.

Case 1: $\mu(L_{\le}) < t\mu(K)$ for sufficiently large constant c_7 and sufficiently large Q.

Let $w \in \sigma_*(P) \cap S_P(\alpha_1)$ for some $P \in \mathcal{P}_n\left(Q^{\frac{d_n-n}{n(n+1)}}\right)$. Then by (45) and Lemma 1 (for $j = n$), we have

$$|w-\alpha_1|_p \le (c_5c_8^{-1}Q^{-d_n})^{1/n}. \tag{47}$$

Summing the estimate (47) over all polynomials $P \in \mathcal{P}_n\left(Q^{\frac{d_n-n}{n(n+1)}}\right)$, we obtain

$$\mu(L_{\le}) \le (2Q^{\frac{d_n-n}{n(n+1)}}+1)^{n+1}c_5^{1/n}c_8^{-1/n}Q^{-d_n/n}n \le t\mu(K)$$

for $c_7 \ge 2^{n+2}nc_5^{1/n}c_8^{-1/n}t^{-1}$ and $Q > Q_0$.

Case 2: $\mu(L_{>}) < 2t\mu(K)$ for sufficiently large Q.

For every irreducible polynomial $P \in \mathcal{P}_n(c_6Q^{r_n}) \setminus \mathcal{P}_n\left(Q^{\frac{d_n-n}{n(n+1)}}\right)$ we define the set

$$A(P) = \{\alpha_1 :\ P(\alpha_1) = 0 \ \text{ and } \ |P'(\alpha_1)|_p \le pc_5^{1/2}Q^{-d_n/2}\}.$$

For $k \in \mathbb{N}$, let $\mathcal{P}_{\mathbf{l}}^k$ denote the subclass of $\mathcal{P}_n(\mathbf{l})$ given by

$$\mathcal{P}_{\mathbf{l}}^k = \{P \in \mathcal{P}_n(\mathbf{l}) :\ 2^{k-1} < H(P) \le 2^k\}.$$

Then we have

$$\mathcal{P}_n(c_6Q^{r_n}) \setminus \mathcal{P}_n\left(Q^{\frac{d_n-n}{n(n+1)}}\right) = \bigcup_{\mathbf{l}} \bigcup_{k=[\frac{d_n-n}{n(n+1)}\log_2 Q]+1}^{[(r_n+\epsilon)\log_2 Q]} \mathcal{P}_{\mathbf{l}}^k$$

for $\epsilon > 0$ and $Q > Q_0$.

Now divide the cylinder K into smaller cylinders J_i' with $\mu(J_i') = c_{23}2^{k(u'+\gamma)}$ where $c_{23} > c_{24}$, $c_{24} = \max_{1\le j\le n}(c_8^{-1}c_6^{d_n/r_n}c_5)^{1/j}$, $\gamma \ge n\epsilon_1$, $r_n(u'+\gamma) \le -1$ and

$$u' = \min_{1\le j\le n}\{(-d_n/r_n + q_j)/j\},\ q_n = 0.$$

Note for $j = n$ from the last estimate we have $u' = -d_n/(nr_n)$. Then from inequality $r_n(u' + \gamma) \leq -1$ we obtain that $\gamma \leq (d_n - n)/(nr_n)$. Choose $\gamma = 1/(2n)$.

First show that the assumption that at least two irreducible polynomials from $\mathcal{P}_1^k$ without common roots belong to the cylinder J_i' will lead to a contradiction. To show this, suppose that P_1 and P_2 belong to J_i'. By Lemma 3 and (46) we have $c(n)H(P)^{-q_1} < |P'(\alpha_1)|_p \leq pc_5^{1/2}Q^{-d_n/2}$, which implies that $q_1 > d_n/(2r_n)$ for $H(P) \leq c_6Q^{r_n}$ and sufficiently large Q. Develop P_1 as a Taylor series in the neighbourhood J_i' of α_1 to obtain

$$|P(w)|_p < 2^{k(-d_n/r_n+(n+1)\gamma)}, \qquad w \in J_i'$$

for sufficiently large k, where

$$\begin{aligned}|(j!)^{-1}P^{(j)}(\alpha_1)|_p|w-\alpha_1|_p^j &< p^j 2^{(k-1)(-q_j+(n-j)\epsilon_1)}c_{23}^j 2^{k(j\gamma+j(\frac{-d_n/r_n+q_j}{j}))} = \\ &= p^j c_{23}^j 2^{q_j-(n-j)\epsilon_1}2^{k(j\gamma-d_n/r_n+(n-j)\epsilon_1)}, \quad 1 \leq j \leq n.\end{aligned}$$

Obviously, the same estimate holds for P_2 on J_i'. Apply Lemma 4 to polynomials P_1 and P_2 with $\tau = d_n/r_n - (n+1)\gamma$ and $\eta = -u' - \gamma - \epsilon_1$. Therefore

$$\begin{aligned}\tau + 2\max(\tau-\eta, 0) &= 3d_n/r_n + 2(-d_n/r_n + q_j)/j - \gamma(3n+1) + 2\epsilon_1 \geq \\ &\geq \begin{cases} 2d_n/r_n - \gamma(3n+1) + 2\epsilon_1, & 2 \leq j \leq n, \\ d_n/r_n + 2q_1 - \gamma(3n+1) + 2\epsilon_1, & j = 1.\end{cases}\end{aligned}$$

Since $q_1 > d_n/(2r_n)$, $d_n \geq n + r_n$ and $r_n \leq 1$, it is readily seen that $\tau + 2\max(\tau - \eta, 0) > 2n + 2 - \gamma(3n+1) + 2\epsilon_1$ in both cases. Since $\gamma = 1/(2n)$ the last inequality gives a contradiction in (8) for $\theta \leq (n-1)/(2n)$.

Therefore, there is at most one irreducible polynomial $P \in \mathcal{P}_1^k$ that belongs to J_i' or there are two irreducible polynomials $P_i \in \mathcal{P}_1^k$, of the form $P_i(w) = \pm P(w)$ for some irreducible polynomial $P \in \mathcal{P}_1^k$, belong to the cylinder J_i'. This will divide the polynomials P into two classes with respect to the cylinder J' : class I and class II respectively. According to this classification, it follows that

$$L_> \subseteq L_I \cup L_{II}$$

where $L_j = \bigcup\limits_{\mathbf{1}} \bigcup\limits_{k=[\frac{d_n-n}{n(n+1)}\log_2 Q]+1}^{[(r_n+\epsilon)\log_2 Q]} \bigcup\limits_{\substack{P\in\mathcal{P}_1^k \\ P \text{ of class } j}} \sigma_*(P)$ for $j = I, II$.

For $P \in \mathcal{P}_1^k$ denote by $\nu(P, \alpha_1)$ the set of $w \in S_P(\alpha_1)$ satisfying (45) and (46). According to Lemma 1 and Lemma 3 we have that

$$\mu(\nu(P,\alpha_1)) < c_{24}2^{ku'}.$$

Using the inclusion $\sigma_*(P) \subseteq \bigcup\limits_{\alpha_1\in A(P)} \nu(P,\alpha_1)$ for any polynomial P and the fact that the number of polynomials $P \in \mathcal{P}_1^k$ of class I does not exceed the number of cylinders

J', we obtain

$$\mu(L_I) \le \sum_{1} \sum_{k=[\frac{dn-n}{n(n+1)}\log_2 Q]+1}^{[(r_n+\epsilon)\log_2 Q]} nc_{23}^{-1}c_{24}2^{ku'}2^{k(-u'-\gamma)}\mu(K) <$$
$$< nC(n,\epsilon_1)c_{23}^{-1}c_{24}\mu(K)\sum_{k=0}^{\infty}2^{-k/(2n)} < nC(n,\epsilon_1)c_{23}^{-1}c_{24}2^{1/(2n)}(2^{1/(2n)}-1)^{-1}\mu(K) < \quad (48)$$
$$< 4n^2C(n,\epsilon_1)c_{23}^{-1}c_{24}\mu(K) \le t\mu(K)$$

for $c_{23} \ge 4n^2t^{-1}c_{24}C(n,\epsilon_1)$ and sufficiently large Q.

It is easy to see that the measure $\mu(L_{II})$ coincides with the measure $\mu(L_I)$. □

3.3 Reducible polynomials

Let $n \ge 3$. Now we need to consider the case

$$|P(w)|_p < c_5Q^{-d_n}, \quad |P'(w)|_p \le c_{20}Q^{-(n-1+r_n)/2}, \quad |P'(\alpha_1)|_p \le c_{20}Q^{-(n-1+r_n)/2}. \quad (49)$$

Define the subset $\mathcal{L}_{red}$ of the set $\bar{\mathcal{L}}_n$ containing $w \in K$ for which there exists reducible polynomial $P \in \mathcal{P}_n(c_6Q^{r_n})$ satisfying (49).

Proposition 8. *For sufficiently large constant c_7 and sufficiently large Q we have $\mu(\mathcal{L}_{red}) < \left(\sum_{k=1}^{[n/2]}(4s(k)+3s(n-k))+n-1\right)t\mu(K)$.*

Proof. Let $P \in \mathcal{P}_n(c_6Q^{r_n})$ be a reducible polynomial which belongs to K. Let P have the form

$$P(w) = P_1(w)P_2(w), \quad \deg P_1 = n_1, \quad \deg P_2 = n - n_1.$$

Assume without loss of generality that $1 \le n_1 \le n/2$.

3.3.1 Polynomials of the form $P(w) = (P_1(w))^s$

Let $n = n_1s$ and $P(w) = (P_1(w))^s$ where $2 \le s \le n$. Therefore, $H(P_1) < 2^{n_1}c_6^{1/s}Q^{r_n/s}$ and

$$|P_1(w)|_p < c_5^{n_1/n}Q^{-d_n/s}. \quad (50)$$

Let $n_1 = 1$. Therefore, $|P_1(w)|_p = |aw+b|_p < c_5^{1/n}Q^{-d_n/n}$ and $H(P_1) < 2c_6^{1/n}Q^{r_n/n}$. By Lemma 6(iii) we have that the measure of such $w \in K$ does not exceed $2t\mu(K)$ for $c_7 \ge 2^3c_6^{1/n}c_5^{1/n}t^{-1}$.

Let $2 \le n_1 \le n/2$. If $|P_1'(w)|_p < \delta_1$ with $\delta_1 = 2^{-n_1^2-2n_1-9}p^{-2}c_6^{-(n_1+1)n_1/n}c_5^{-n_1/n}t^2$, then by inductive hypothesis the measure of $w \in K$ satisfying (50) does not exceed $s(n_1)t\mu(K)$ for sufficiently large Q. If $|P_1'(w)|_p \ge \delta_1$ then by Lemma 1 we have

$$|w-\alpha_1|_p < c_5^{n_1/n}\delta_1^{-1}Q^{-d_n/s}, \quad w \in S_P(\alpha_1).$$

Summing the last estimate over all polynomials $P_1 \in \mathcal{P}_{n_1}(2^{n_1} c_6^{n_1/n} Q^{r_n/s})$, we obtain that the measure of $w \in K$ satisfying (50) does not exceed $nc_5^{n_1/n} Q^{-d_n/s} \delta_1^{-1}(2^{n_1+1} c_6^{n_1/n} Q^{r_n/s} + 1)^{n_1+1}$, which is less or equal to

$$2^{(n_1+2)(n_1+1)} n \delta_1^{-1} c_6^{n_1(n_1+1)/n} c_5^{n_1/n} Q^{(-n+r_n n_1)n_1/n} \le t\mu(K)$$

for $n_1 \ge 2$, sufficiently large c_7 and sufficiently large Q.

Therefore, further we can assume that $P(w) = P_1(w)P_2(w)$ where P_1 and P_2 does not have common roots.

3.3.2 Polynomials $P(w) = P_1(w)P_2(w)$ where P_1 and P_2 without common roots

For P the set of $w \in K \cap S_P(\alpha_1)$ such that $|P(w)|_p < c_5 Q^{-d_n}$ we denote by $\lambda(P)$. By Gelfond's lemma [11],

$$2^{-n} H(P_1)H(P_2) < H(P) < 2^n H(P_1)H(P_2).$$

Let $c_{25} Q^{r_{n_1}} < H(P_1) \le Q^{r_{n_1}}$, $c_{25} < 1$. Therefore, $H(P_2) < 2^n c_6 c_{25}^{-1} Q^{r_n - r_{n_1}}$. By the continuity of P there exists $a \in \mathbb{R}$ such that

$$\mu\left(w \in \lambda(P) :\ |P_1(w)|_p < c_5 Q^{-a}\right) = \mu(\lambda(P))/2. \tag{51}$$

Then for the complement to (51) we have

$$\mu\left(w \in \lambda(P) :\ |P_1(w)|_p \ge c_5 Q^{-a}\right) = \mu(\lambda(P))/2 \tag{52}$$

or

$$\mu\left(w \in \lambda(P) :\ |P_2(w)|_p < Q^{-d_n+a}\right) = \mu(\lambda(P))/2. \tag{53}$$

Then according to Lemma 5 and by (51), (53) for all $w \in \lambda(P)$ we have

$$|P_1(w)|_p < (2p(n_1+1))^{n_1+1} c_5 Q^{-a},\ |P_2(w)|_p < (2p(n-n_1+1))^{n-n_1+1} Q^{-d_n+a}. \tag{54}$$

For $a > n_1 + r_{n_1}$ we have

$$|P_1(w)|_p < (2p(n_1+1))^{n_1+1} c_5 Q^{-a}, \quad c_{25} Q^{r_{n_1}} < H(P_1) \le Q^{r_{n_1}}. \tag{55}$$

Then by inductive hypothesis, we obtain that the measure of $w \in K$ for which there is the polynomial $P(w) = P_1(w)P_2(w)$ with P_1 satisfying (55) does not exceed $s(n_1)t\mu(K)$ for sufficiently large Q.

For $a < n_1 + r_{n_1}$ we have

$$|P_2(w)|_p < (2p(n-n_1+1))^{n-n_1+1} Q^{-d_n+a},\ \ H(P_2) < 2^n c_6 c_{25}^{-1} Q^{r_n - r_{n_1}}. \tag{56}$$

Then by inductive hypothesis, we obtain that the measure of $w \in K$ for which there is the polynomial $P = P_1P_2$ with P_2 satisfying (56) does not exceed $s(n-n_1)t\mu(K)$ for sufficiently large Q.

Further we consider the case when $a = n_1+r_{n_1}$. By (49) we have that $|P'(\alpha_1)|_p$ takes the small value. Therefore, there exist l, $2 \leq l \leq n$, roots of P which are close to each other. Let $\delta_2 \in \mathbb{R}^+$ which we specify later. Since α_1 is the nearest root to $w \in \lambda(P)$, reorder the other roots of P so that

$$|\alpha_1 - \alpha_2|_p \leq \ldots \leq |\alpha_1 - \alpha_l|_p < \delta_2 \leq |\alpha_1 - \alpha_{l+1}|_p \leq \ldots \leq |\alpha_1 - \alpha_n|_p,\ 2 \leq l \leq n.$$

From $P'(\alpha_1) = a_n(\alpha_1 - \alpha_2)\ldots(\alpha_1 - \alpha_n)$ and (49) we have

$$|\alpha_1 - \alpha_2|_p|\alpha_1 - \alpha_3|_p \ldots |\alpha_1 - \alpha_l|_p < c_8^{-1}c_{20}Q^{-(n-1+r_n)/2}\delta_2^{-(n-l).} \tag{57}$$

Case 1. If $l \geq 3$ then there exist at least two roots of the polynomial P which belong to one of the polynomials P_1 or P_2; say that α_2 and α_3 are the roots of P_1. From (5) it follows that the roots of P are bounded, i.e. $|\alpha_i|_p < c_{26}$, $1 \leq i \leq n$. Then

$$|P_1'(\alpha_2)|_p = |a_{n_1}(P_1)(\alpha_2 - \alpha_3) \prod_{3\leq s\leq n_1} (\alpha_2 - \alpha_s')|_p < \delta_2 c_{26}^{n_1-2}, \tag{58}$$

where $P_1(\alpha_s') = 0$. Since $w \in S_P(\alpha_1)$ then, using Lemma 1, we have

$$|w - \alpha_2|_p \leq \max(|w - \alpha_1|_p, |\alpha_1 - \alpha_2|_p) = \max((c_5Q^{-d_n})^{1/n}, \delta_2) = \delta_2 \tag{59}$$

for $Q > Q_0$. By (58) and (59), we get $|P_1'(w)|_p = \left|\sum_{i=1}^{n_1}((i-1)!)^{-1}P_1^{(i)}(\alpha_2)(w - \alpha_2)^{i-1}\right|_p <$ $\delta_2 \max(1, c_{26}^{n_1-2})$. Thus, we have

$$\begin{gathered} |P_1(w)|_p < (2p(n_1+1))^{n_1+1}c_5Q^{-(n_1+r_{n_1})}, \qquad c_{25}Q^{r_{n_1}} < H(P_1) \leq Q^{r_{n_1}}, \\ |P_1'(w)|_p < \delta_2 \max(1, c_{26}^{n_1-2}). \end{gathered} \tag{60}$$

Choose $\delta_2 \leq 2^{-2n_1-10}p^{-n_1-3}(n_1+1)^{-(n_1+1)}c_5^{-1}(\max(1, c_{26}^{n_1-2}))^{-1}t^2$. Then by inductive hypothesis, we obtain that the measure of $w \in K$ for which there is the polynomial $P = P_1P_2$ with P_1 satisfying (60) does not exceed $s(n_1)t\mu(K)$ for sufficiently large c_7 and sufficiently large Q.

If at least two roots of P belong to P_2 then similarly we obtain that the measure of $w \in K$ does not exceed $s(n - n_1)t\mu(K)$ for $Q > Q_0$ and sufficiently large c_7.

Case 2. Let $l = 2$. If α_1 and α_2 belong to one polynomial P_1 or P_2 then the proof is coincided with the Case 1. Now assume without loss of generality that α_1 is a root of P_1 and α_2 is a root of P_2. In this case for any distinct roots of the polynomials P_1 and P_2 we have $|\alpha_{i_1}(P_j) - \alpha_{i_2}(P_j)|_p \geq \delta_2$. Thus,

$$|P_1'(\alpha_1)|_p > c_{27}\delta_2^{(n_1-1)}, \qquad |P_2'(\alpha_2)|_p > c_{28}\delta_2^{(n-n_1-1)}. \tag{61}$$

Consider the resultant of the polynomials P_1 and P_2 which have no common roots:

$$R(P_1, P_2) = a_{n_1}^{n-n_1}(P_1)a_{n-n_1}^{n_1}(P_2)(\alpha_1 - \alpha_2) \prod_{\substack{1\leq i\leq n_1,\, 1\leq j\leq n-n_1, \\ \alpha_i'\neq\alpha_1,\, \alpha_j''\neq\alpha_2}} (\alpha_i' - \alpha_j''),$$

where $P_1(\alpha_i')=0$ and $P_2(\alpha_j'')=0$. From (57) we have

$$|\alpha_1-\alpha_2|_p < c_8^{-1}c_{20}Q^{-(n-1+r_n)/2}\delta_2^{-(n-2)}.$$

Using the fact that the roots of P are bounded and the estimate

$$|a_{n_1}^{n-n_1}(P_1)a_{n-n_1}^{n_1}(P_2)| < Q^{r_{n_1}(n-n_1)}(2^n c_6 c_{25}^{-1}Q^{r_n-r_{n_1}})^{n_1},$$

we get

$$\begin{aligned} 2^{-nn_1}c_6^{-n_1}c_{25}^{n_1}Q^{-r_{n_1}(n-n_1)+n_1(-r_n+r_{n_1})} &\le |R(P_1,P_2)|_p, \\ |R(P_1,P_2)|_p &< c_8^{-1}c_{20}c_{26}^{n_1(n-n_1)-1}\delta_2^{-(n-2)}Q^{-(n-1+r_n)/2}. \end{aligned} \tag{62}$$

We have a contradiction in (62) for sufficiently small c_{20} and $r_n \le 1$ if $n=2n_1$ and $r_{n_1} \le \frac{n-1+r_n-2n_1r_n}{2(n-2n_1)}$ if $n>2n_1$.

Now we are left with the case when

$$r_{n_1} > \frac{n-1+r_n-2n_1r_n}{2(n-2n_1)} \tag{63}$$

with $1\le n_1 < n/2$. For P_2 we have

$$|P_2(w)|_p < (2p(n-n_1+1))^{n-n_1+1}Q^{-d_n+n_1+r_{n_1}}, \quad P_2\in\mathcal{P}_{n-n_1}(2^n c_6 c_{25}^{-1}Q^{r_n-r_{n_1}}). \tag{64}$$

By (54), (61) and Lemma 1, we have that

$$|w-\alpha_2|_p < (2p(n-n_1+1))^{n-n_1+1}c_{28}^{-1}\delta_2^{-(n-n_1-1)}Q^{-d_n+n_1+r_{n_1}+(r_n-r_{n_1})}$$

for $w\in S_{P_2}(\alpha_2)$. Summing the last estimate over all polynomials

$$P_2\in\mathcal{P}_{n-n_1}(2^n c_6 c_{25}^{-1}Q^{r_n-r_{n_1}})$$

and using (63), we obtain that the measure of $w\in K$ for which there is the polynomial $P=P_1P_2$ with P_2 satisfying (64) does not exceed

$$c_{29}Q^{-n+n_1+r_n(n-n_1)-r_{n_1}(n-n_1)} < t\mu(K)$$

for sufficiently large Q. $\square$

Combining all estimates, starting from Proposition 1, we obtain that the measure of $\bar{\mathcal{L}}_n$ does not exceed $s(n)t\mu(K)$ with

$$s(n)=2n+13+\sum_{k=3}^{n-1}s(k)+\sum_{k=1}^{[n/2]}(4s(k)+3s(n-k)) \quad \text{for } n\ge 3, \tag{65}$$

$s(1)=2$ and $s(2)=14$. Choose $t=l\cdot(s(n))^{-1}$.

Finally, we turn to the proof of Theorem 1.

4 Proof of Theorem 1

Let $\delta_0 \in \mathbb{R}^+$. Consider the set $\bar{\mathcal{L}}_n(Q, \delta_0, K)$ with $d_n = n + 1$. By Theorem 2 there exists a constant δ_0, which satisfies the following property: for any cylinder K in K_0 there exists a sufficiently large number $Q_0 = Q_0(K)$ such that for $\mu(K) > c_7 Q_0^{-1}$ and sufficiently large constant c_7, which does not depend on Q_0, and for all $Q > Q_0$ we have $\mu(\bar{\mathcal{L}}_n(Q, \delta_0, K)) < l\mu(K)$. For the rest of the proof we may assume that c_7 is a constant which is greater or equal to $\frac{2\cdot 3^n}{(1-l)\delta_0}$ and for which Theorem 2 is valid.

Denote by $\mathcal{L}_0(Q, K)$ the set of $w \in K$, for which the inequality $|P(w)|_p < Q^{-(n+1)}$ is satisfied for some $P \in \mathcal{P}_n(Q)$. It can be readily verified using Dirichlet's box principle that $\mathcal{L}_0(Q, K) = K$. By Theorem 2 there exists a set $\mathcal{L}_n(Q, \delta_0, K) = K \setminus \bar{\mathcal{L}}_n(Q, \delta_0, K) \subset K$ such that $\mu(\mathcal{L}_n(Q, \delta_0, K)) \geq (1 - l)\mu(K)$ for all $Q > Q_0$, where $Q_0 > c_7\mu(K)^{-1}$.

Denote by $\mathcal{L}_{\leq(n-1)}(Q, \delta_0, K)$ the union of the cylinders $\sigma(\alpha) = \{w \in K : |w - \alpha|_p < \delta_0^{-1}Q^{-(n+1)}\}$ over all algebraic numbers in $\mathbb{Z}_p$ of degree at most $n-1$ and height at most Q. The number of different cylinders in this union is at most $(2Q+1)^n$ and every cylinder has a measure at most $\delta_0^{-1}Q^{-(n+1)}$, therefore we conclude that $\mu(\mathcal{L}_{\leq(n-1)}(Q, \delta_0, K)) \leq (1 - l)\mu(K)/2$ for $c_7 \geq \frac{2\cdot 3^n}{(1-l)\delta_0}$.

Let $\mathcal{L}'_n(Q, \delta_0, K)$ be defined by

$$\mathcal{L}'_n(Q, \delta_0, K) = \mathcal{L}_n(Q, \delta_0, K) \setminus \mathcal{L}_{\leq(n-1)}(Q, \delta_0, K).$$

Let $w \in \mathcal{L}'_n(Q, \delta_0, K)$. Then by Hensel's Lemma [17] there is a root $\alpha \in \mathbb{Z}_p$ of P such that

$$|w - \alpha|_p < \delta_0^{-1}Q^{-(n+1)}. \tag{66}$$

If Q is sufficiently large then $\alpha \in K$. Since $w \notin \mathcal{L}_{\leq(n-1)}(Q, \delta_0, K)$ then we conclude that the degree of α is exactly n.

Choose the maximal collection $\{\alpha_1, \ldots, \alpha_{\mathbf{t}}\}$ of algebraic numbers in $K \cap \mathcal{A}_{n,p}$ satisfying

$$H(\alpha_i) \leq Q, \quad |\alpha_i - \alpha_j|_p \geq Q^{-(n+1)}, \quad 1 \leq i < j \leq \mathbf{t}.$$

Since the collection $\{\alpha_1, \ldots, \alpha_{\mathbf{t}}\}$ is maximal then there exists α_i in this collection such that $|\alpha - \alpha_i|_p \leq Q^{-(n+1)}$. From this and (66) it follows that $|w - \alpha_i|_p < \delta_0^{-1}Q^{-(n+1)}$. As w is an arbitrary point of $\mathcal{L}'_n(Q, \delta_0, K)$ then

$$\mathcal{L}'_n(Q, \delta_0, K) \subset \bigcup_{i=1}^{\mathbf{t}} \{w \in K : |w - \alpha_i|_p < \delta_0^{-1}Q^{-(n+1)}\}.$$

Since $\mu(\mathcal{L}'_n(Q, \delta_0, K)) \geq (1 - l)\mu(K)/2$, we have $\mathbf{t} \gg Q^{n+1}\mu(K)$. Let $T = Q^{n+1}$ then for any $T \geq T_0$, where $T_0 = (c_7+1)^{n+1}\mu(K)^{-(n+1)}$, there exists a collection $\alpha_1, \ldots, \alpha_{\mathbf{t}} \in K \cap \mathcal{A}_{n,p}$ satisfying (1) which completes the proof of the theorem.

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АННОТАЦИЯ

В данной статье мы доказываем, что для достаточно больших чисел $Q \in \mathbb{N}$ существуют цилиндры $K \subset \mathbb{Q}_p$ с мерой Хаара $\mu(K) \leq \frac{1}{2}Q^{-1}$, которые не содержат алгебраических p-адических чисел α степени $\deg \alpha = n$ и высоты $H(\alpha) \leq Q$. Основной результат показывает, что в любом цилиндре K, $\mu(K) > c_1 Q^{-1}$, $c_1 > c_0(n)$, существует не менее $c_3 Q^{n+1}\mu(K)$ алгебраических p-адических чисел $\alpha \in K$ степени n и $H(\alpha) \leq Q$.

Ключевые слова: *целочисленные многочлены, алгебраические p-адические числа, регулярная система, мера Хаара.*