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On transition functions of Topological Toric Manifolds

We show that any topological toric manifold can be covered by finitely many open charts so that all the transition functions between these charts are Laurent monomials of z_j 's and \bar{z}_j 's. In addition, we will describe toric manifolds and some special class of topological toric manifolds in terms of transition functions of charts up to (weakly) equivariant diffeomorphism.

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Key words: *toric manifold, quasitoric manifold, Laurent monomial.*

1. Background

The topological toric manifold was introduced as an analogue of toric manifolds (i.e. compact smooth toric varieties) in the category of closed smooth manifolds by H. Ishida, Y. Fukukawa and M. Masuda, [1]. By definition, a *topological toric manifold* X is a closed smooth manifold of dimension $2n$ with an effective smooth action of $(\mathbb{C}^*)^n$ so that X has an open dense orbit, and X is covered by finitely many $(\mathbb{C}^*)^n$ -invariant open subsets each of which is equivariantly diffeomorphic to a smooth representation space of $(\mathbb{C}^*)^n$. Since the $(\mathbb{C}^*)^n$ -action is effective, the smooth representation of $(\mathbb{C}^*)^n$ in each invariant open subset must be faithful, hence it is isomorphic to a direct sum of complex one-dimensional smooth (linear) representation spaces of $(\mathbb{C}^*)^n$. Notice that there is a *canonical* $(S^1)^n$ -action on X by restricting the $(\mathbb{C}^*)^n$ -action to the standard compact torus $(S^1)^n \subset (\mathbb{C}^*)^n$. In this paper, we will always assume that a topological toric manifold X is equipped with this $(S^1)^n$ -action too.

It is shown in section 7 of [1] that a topological toric manifold X is always simply connected. The orbit space of the $(S^1)^n$ -action on X is a nice manifold with corners whose faces (including the whole orbit space) are all contractible and, any intersection of faces is either connected or empty. So the orbit space looks like a simple polytope. In addition, similar to toric manifolds, the integral cohomology ring $H^*(X)$ is generated by the elements of $H^2(X)$.

In a $2n$ -dimensional topological toric manifold X , there exist finitely many codimension-two closed connected submanifolds X_1, \dots, X_m in X each of which is fixed pointwise by some \mathbb{C}^* -subgroup of $(\mathbb{C}^*)^n$. In fact, the Poincaré dual to X_1, \dots, X_m forms an integral basis of $H^2(X)$. Such X_1, \dots, X_m are called the *characteristic submanifolds* of X . The choice of orientation on each characteristic submanifold X_i together with an orientation on X is called an *omniorientation* on X . From X_1, \dots, X_m , we can define an abstract simplicial complex $\Sigma(X)$ of dimension $n - 1$

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(with the empty set \emptyset added) as following.

$$\Sigma(X) := \{I \subset [m] \mid X_I := \bigcap_{i \in I} X_i \neq \emptyset\} \cup \{\emptyset\} \quad (1)$$

where $[m]$ denote the set $\{1, \dots, m\}$. $\Sigma(X)$ is a *pure* simplicial complex in the sense that any simplex in $\Sigma(X)$ is contained in some simplex of maximal dimension $(n - 1)$.

Let $\Sigma^{(k)}(X)$ be the set of $(k - 1)$ -simplices in $\Sigma(X)$. Then $\Sigma^{(1)}(X)$ can be identified with $[m]$. It is shown by Lemma 3.6 in [1] that X_1, \dots, X_m intersect transversely with each other and any X_I ($I \subset [m]$) is a closed connected submanifold of dimension $2(n - |I|)$ in X . The characteristic submanifolds of X play a similar role as invariant irreducible divisors of a toric variety.

The major difference between topological toric manifolds and toric manifolds is that the local action of $(\mathbb{C}^*)^n$ is equivariantly diffeomorphic to a smooth vs. algebraic linear representation space of $(\mathbb{C}^*)^n$. For example when $n = 1$, any smooth complex one-dimensional representation of $\mathbb{C}^* = \mathbb{R}_{>0} \times S^1$ corresponds to a smooth endomorphism of \mathbb{C}^* , which can be written in the following form:

$$g \longmapsto |g|^{b+\sqrt{-1}c} \left(\frac{g}{|g|}\right)^v, \text{ where } (b + \sqrt{-1}c, v) \in \mathbb{C} \times \mathbb{Z}. \quad (2)$$

A smooth endomorphism of \mathbb{C}^* in (2) is algebraic if and only if $b = v$ and $c = 0$. So the group $\text{Hom}(\mathbb{C}^*, \mathbb{C}^*)$ of *smooth endomorphisms* of \mathbb{C}^* is isomorphic to $\mathbb{C} \times \mathbb{Z}$, while the group $\text{Hom}_{\text{alg}}(\mathbb{C}^*, \mathbb{C}^*)$ of *algebraic endomorphisms* of \mathbb{C}^* is isomorphic to \mathbb{Z} . Hence there are much more smooth linear representations of $(\mathbb{C}^*)^n$ than algebraic ones. And so the family of topological toric manifolds is much larger than the family of toric manifolds.

There is another topological analogue of toric manifold introduced by Davis and Januszkiewicz [2] in the early 1990s, now called “quasitoric manifold” (see [3]). A *quasitoric manifold* is a $2n$ -dimensional smooth manifold with a locally standard $(S^1)^n$ -action whose orbit space is a simple convex polytope. It is shown in section 10 of [1] that any quasitoric manifold has the structure of a topological toric manifold. In fact, there are uncountably many topological toric manifold structures for any given quasitoric manifold M . But there is no canonical choice of such a structure for M .

It is shown in section 10 of [1] that the family of $2n$ -dimensional topological toric manifolds with the canonical $(S^1)^n$ -actions is strictly larger than the family of $2n$ -dimensional quasitoric manifolds up to equivariant homeomorphisms. The relations between toric manifolds, quasitoric manifolds and topological toric manifolds can be explained by the diagram in Figure 1 (see chapter 5 of [3]).

In this paper, we will study topological toric manifolds from the viewpoint of transition functions of charts. The paper is organized as follows. In section 2, we will review the basic construction in [1] of a topological toric manifold $X(\Delta)$ from a complete non-singular topological fan Δ , and define a set of open charts called *normal charts* which cover the whole $X(\Delta)$. A very special property of these charts is that the transition functions between any two normal charts are completely determined by the topological fan Δ . This allows us to give an equivalent description of toric manifolds in terms of transition functions of some charts up to (weakly) equivariant diffeomorphism (see Theorem 2.3). In section 3, we will show that any topological toric manifold can be covered by finitely many $(\mathbb{C}^*)^n$ -invariant open charts so that all the transition functions are Laurent monomials of z_j 's and \bar{z}_j 's (see Corollary 3.6). There should be some special geometrical properties on a topological toric manifold implied by the existence of such an atlas. It is interesting to see what these geometric properties are. In section 4, we will define the notion of a *nice topological toric manifold*. Similar to toric manifolds, we can also describe nice topological toric manifolds in terms of transition functions of charts up to (weakly) equivariant diffeomorphism (see Theorem 4.3).

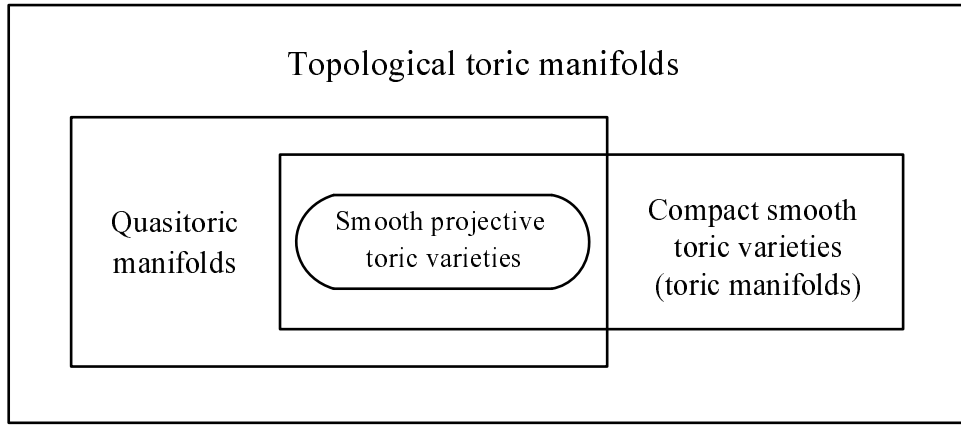


Figure 1.

In this paper, we will quote many lemmas and theorems from [1]. But we will avoid repeating the full statements of those lemmas and theorems. Instead, we will only indicate where they are in [1]. So the reader should get familiar with the content of [1] before reading the arguments in this paper.

2. Topological fan, transition functions and Normal charts

Suppose a $2n$ -dimensional topological toric manifold X is equipped with an orientation. Using the notations in the discussion in section 1, for each characteristic submanifold X_i in X , $i \in [m] = \Sigma^{(1)}(X)$, the \mathbb{C}^* -subgroup of $(\mathbb{C}^*)^n$ which fixes X_i determines a unique element $\lambda_{\beta_i(X)} \in \text{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ where $\beta_i(X) \in \mathbb{C}^n \times \mathbb{Z}^n$ (see Lemma 3.3 in [1]). So we have a map

$$\beta(X) : \Sigma^{(1)}(X) = [m] \rightarrow \mathbb{C}^n \times \mathbb{Z}^n, \quad \beta(X)(i) = \beta_i(X) \text{ for } \forall i \in [m].$$

It is shown by Lemma 3.8 in [1] that $\Delta(X) := (\Sigma(X), \beta(X))$ is a complete non-singular *topological fan* of dimension n . Roughly speaking, a topological fan consists of two simplicial fans which encode the information of the \mathbb{C}^n and \mathbb{Z}^n components of $\beta(X)$, respectively (see section 3 of [1] for the precise definition).

Conversely, Theorem 8.1 in [1] shows that any complete non-singular topological fan $\Delta = (\Sigma, \beta)$ of dimension n determines an oriented topological toric manifold $X(\Delta)$ of dimension $2n$ up to orientation-preserving equivariant diffeomorphism. Indeed, $X(\Delta)$ is defined to be the following quotient space

$$X(\Delta) := U(\Sigma)/\text{Ker}(\lambda_\beta) \tag{3}$$

where $U(\Sigma)$ is an open subspace of \mathbb{C}^m which depends only on Σ and $m = |\Sigma^{(1)}|$ is the number of 0-simplices in Σ . More precisely,

$$U(\Sigma) := \bigcup_{I \in \Sigma} U(I) \tag{4}$$

$$U(I) := \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_i \neq 0 \text{ for } \forall i \notin I\}.$$

In (3), $\lambda_\beta : (\mathbb{C}^*)^m \rightarrow (\mathbb{C}^*)^n$ is a surjective group homomorphism determined by β . The natural action of $(\mathbb{C}^*)^m$ on \mathbb{C}^m leaves $U(\Sigma)$ invariant, so it induces an effective action of $(\mathbb{C}^*)^m/\text{Ker}(\lambda_\beta)$ on $X(\Delta)$ which has an open dense orbit. Since λ_β is surjective, $(\mathbb{C}^*)^m/\text{Ker}(\lambda_\beta)$ is isomorphic to

$(\mathbb{C}^*)^n$. In this way, $X(\Delta)$ get a $(\mathbb{C}^*)^n$ -action with an open dense orbit. Moreover, if we restrict λ_β to the standard compact torus $(S^1)^m \subset (\mathbb{C}^*)^m$, we get a surjective group homomorphism

$$\widehat{\lambda}_\beta : (S^1)^m \rightarrow (S^1)^n \subset (\mathbb{C}^*)^n. \quad (5)$$

If we write $\beta_i = (b_i + \sqrt{-1}c_i, v_i) \in \mathbb{C}^n \times \mathbb{Z}^n$, $i \in [m] = \Sigma^{(1)}$, then $\widehat{\lambda}_\beta$ is determined only by the component $\{v_i\}_{i \in [m]}$ (see section 4 of [1]). It is clear that

$$\text{Ker}(\widehat{\lambda}_\beta) = \text{Ker}(\lambda_\beta) \cap (S^1)^m \subset (\mathbb{C}^*)^m. \quad (6)$$

The correspondence between ominioriented topological toric manifolds and complete non-singular topological fans generalizes the classical correspondence between toric manifolds and complete non-singular fans (see [4] or [5]).

Definition 2.1 (Weakly Equivariant Diffeomorphism). For a Lie group G , two smooth G -manifolds M and N are called *weakly equivariantly diffeomorphic via a diffeomorphism* $F : M \rightarrow N$ if there exists an automorphism σ of G so that

$$F(g \cdot x) = \sigma(g) \cdot F(x) \text{ for any } g \in G \text{ and any } x \in M.$$

And we call F a *weakly equivariant diffeomorphism* between M and N . If σ can be taken to be the identity map of G , then M and N are called *equivariantly diffeomorphic* and F is called an *equivariant diffeomorphism*.

Similarly, we can define the notation of (weakly) equivariant homeomorphism by replacing “diffeomorphism” by “homeomorphism” in the above definition.

It is shown in section 3 of [1] that there are three different levels of equivalence relations among complete non-singular topological fans which give isomorphism, equivariant diffeomorphism and equivariant homeomorphism among the corresponding ominioriented topological toric manifolds, respectively (see Lemma 3.9 in [1]).

From any complete non-singular topological fan $\Delta = (\Sigma, \beta)$ of dimension n , we can define an atlas \mathcal{U} on the ominioriented topological toric manifold $X(\Delta)$ whose charts are indexed by all the $(n-1)$ -simplices $I \in \Sigma^{(n)}$. Indeed, the open chart in \mathcal{U} corresponding to an $I \in \Sigma^{(n)}$ is

$$\varphi_I : V_I = U(I)/\text{Ker}(\lambda_\beta) \rightarrow \mathbb{C}^n. \quad (7)$$

It is easy to see that each $V_I \subset X(\Delta)$ is *invariant* under the $(\mathbb{C}^*)^n$ -action on $X(\Delta)$, i.e. $g(V_I) \subset V_I$ for any $g \in (\mathbb{C}^*)^n$ (see section 4 of [1]). We call each $\varphi_I : V_I \rightarrow \mathbb{C}^n$ a *normal chart* of $X(\Delta)$. It is clear that $X(\Delta)$ is covered by finitely many normal charts $\{\varphi_I : V_I \rightarrow \mathbb{C}^n, I \in \Sigma^{(n)}\}$. When we observe the $(\mathbb{C}^*)^n$ -action on $X(\Delta)$ through any normal chart $\varphi_I : V_I \rightarrow \mathbb{C}^n$, the $(\mathbb{C}^*)^n$ -action is a faithful smooth linear representation. In other words, the $(\mathbb{C}^*)^n$ -action in V_I is equivariantly homeomorphic to a faithful smooth linear representation of $(\mathbb{C}^*)^n$ in \mathbb{C}^n via the map φ_I .

Next, let us see what the characteristic submanifolds of $X(\Delta)$ are. We define

$$U(\Sigma)_i := U(\Sigma) \cap \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_i = 0\}, \quad i \in \Sigma^{(1)}. \quad (8)$$

Let $\Phi_\Delta : U(\Sigma) \rightarrow U(\Sigma)/\text{Ker}(\lambda_\beta) = X(\Delta)$ denote the quotient map. Then all the characteristic submanifolds of $X(\Delta)$ are

$$X(\Delta)_i = \Phi_\Delta(U(\Sigma)_i), \quad i \in \Sigma^{(1)}.$$

Observe that for any $I \in \Sigma^{(n)}$, we have

$$U(I) = U(\Sigma) - \bigcup_{i \notin I} U(\Sigma)_i, \quad (9)$$

$$\text{and so } V_I = \Phi_\Delta(U(I)) = X(\Delta) - \bigcup_{i \notin I} X(\Delta)_i. \quad (10)$$

So each open subset V_I is the complement of a set of characteristic submanifolds $\{X(\Delta)_i, i \notin I\}$ in $X(\Delta)$.

Remark 2.2. If we consider $X(\Delta)$ as a $(S^1)^n$ -manifold, we can similarly define the notion of characteristic submanifold of $X(\Delta)$ with respect to the $(S^1)^n$ -action, which are submanifolds fixed pointwise by some S^1 -subgroup of $(S^1)^n$. Indeed, this notion has been used in the study of quasitoric manifolds in [2] and [3]. It is easy to see that the characteristic submanifolds of $X(\Delta)$ with respect to the $(\mathbb{C}^*)^n$ -action and the canonical $(S^1)^n$ -action actually coincide.

The transition functions between any two normal charts $\varphi_I : V_I \rightarrow \mathbb{C}^n$ and $\varphi_J : V_J \rightarrow \mathbb{C}^n$ are computed in Lemma 5.2 in [1] via the information of β . Let

$$\beta_i = (b_i + \sqrt{-1}c_i, v_i) \in \mathbb{C}^n \times \mathbb{Z}^n, \quad i \in \Sigma^{(1)}. \quad (11)$$

Here we consider $b_i, c_i \in \mathbb{R}^n$ and $v_i \in \mathbb{Z}^n$ as column vectors. The following are some easy consequences of Lemma 5.2 in [1].

- If $b_i = v_i$ and $c_i = 0$ for all $i \in \Sigma^{(1)}$, then $X(\Delta)$ is a toric manifold. In this case, the transition functions between any two charts in \mathcal{U} is of the form

$$\omega_j = f_j(z_1, \dots, z_n), \quad 1 \leq j \leq n$$

where each $f_j(z_1, \dots, z_n)$ is a Laurent monomial in complex variables z_1, \dots, z_n .

Conversely, Suppose M^{2n} is a $2n$ -dimensional closed smooth manifold with an effective smooth action of $(\mathbb{C}^*)^n$ having an open dense orbit. And we assume

- (i) M^{2n} is covered by finitely many $(\mathbb{C}^*)^n$ -invariant open charts $\{\phi_j : V_j \rightarrow \mathbb{C}^n\}_{1 \leq j \leq r}$ with all the transition functions being Laurent monomials of z_1, \dots, z_n ,
- (ii) In one chart $\phi_j : V_j \rightarrow \mathbb{C}^n$, the $(\mathbb{C}^*)^n$ -action in V_j is (weakly) equivariantly homeomorphic to an algebraic linear representation of $(\mathbb{C}^*)^n$ in \mathbb{C}^n via the map ϕ_j .

Then since the transition functions of $\{\phi_j : V_j \rightarrow \mathbb{C}^n\}_{1 \leq j \leq r}$ are all Laurent monomials in z_1, \dots, z_n , it is easy to see that for any other chart $\phi_{j'} : V_{j'} \rightarrow \mathbb{C}^n$, the $(\mathbb{C}^*)^n$ -action in $V_{j'}$ is also (weakly) equivariantly homeomorphic to an algebraic linear representation of $(\mathbb{C}^*)^n$ in \mathbb{C}^n via the map $\phi_{j'}$. Hence M^{2n} is (weakly) equivariantly diffeomorphic to a toric manifold. So we can describe toric manifolds up to (weakly) equivariant diffeomorphism in terms of transition functions of charts as follows.

Theorem 2.3. *Suppose M^{2n} is a $2n$ -dimensional closed smooth manifold with an effective smooth action of $(\mathbb{C}^*)^n$ having an open dense orbit. Then M^{2n} is (weakly) equivariantly diffeomorphic to a toric manifold if and only if M^{2n} can be covered by finitely many $(\mathbb{C}^*)^n$ -invariant open charts $\{\phi_j : V_j \rightarrow \mathbb{C}^n\}_{1 \leq j \leq r}$ so that all the transition functions between these charts are Laurent monomials of z_1, \dots, z_n and, for at least one chart $\phi_j : V_j \rightarrow \mathbb{C}^n$, the $(\mathbb{C}^*)^n$ -action in V_j is (weakly) equivariantly homeomorphic to an algebraic linear representation of $(\mathbb{C}^*)^n$ in \mathbb{C}^n via the map ϕ_j .*

- If b_i is an integral vector congruent to v_i modulo 2 and $c_i = 0$ for all $i \in \Sigma^{(1)}$, then the transition functions between any two charts in \mathcal{U} has the form

$$\omega_j = f_j(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n), \quad 1 \leq j \leq n$$

where each $f_j(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$ is a Laurent monomial of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$,

$$\text{i.e. } f_j(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) = \prod_{i=1}^n z_i^{p_{ij}} \bar{z}_i^{q_{ij}}, \text{ where } p_{ij}, q_{ij} \in \mathbb{Z}.$$

Definition 2.4 (Nice Topological Fan). A complete non-singular topological fan $\Delta = (\Sigma, \beta)$ of dimension n is called *nice* if for each $i \in \Sigma^{(1)}$, we have

$$\beta_i = (b_i + \sqrt{-1} \cdot 0, v_i) \in \mathbb{C}^n \times \mathbb{Z}^n \text{ where } b_i \in \mathbb{Z}^n \text{ and } b_i \equiv v_i \pmod{2}.$$

From the above discussions, for any nice topological fan $\Delta = (\Sigma, \beta)$ of dimension n , the corresponding topological toric manifold $X(\Delta)$ can be covered by finitely many $(\mathbb{C}^*)^n$ -invariant open charts $\{\varphi_I : V_I \rightarrow \mathbb{C}^n\}_{I \in \Sigma^{(n)}}$ with all the transition functions being Laurent monomials of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$.

In section 5 of [1], an explicit ominioriented topological toric manifold structure on $\mathbb{C}P^2 \# \mathbb{C}P^2$ is constructed. The corresponding topological fan is nice. But $\mathbb{C}P^2 \# \mathbb{C}P^2$ is not a toric manifold. So the family of topological toric manifolds is strictly bigger than the family of toric manifolds. In addition, it is natural to ask the following question.

Question: Among all ominioriented topological toric manifolds, how many of them have nice topological fans?

An answer to this question will be given by Theorem 3.4 and Theorem 3.5 in the next section.

3. Transition functions of Topological toric manifolds

According to the discussion in section 2, an ominioriented topological toric manifold with a nice topological fan can be covered by finitely many open charts with all the transition functions being Laurent monomials of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$. In this section, we will show that the topological fan of any ominioriented topological toric manifold can be turned into a nice topological fan via a regular deformation defined as following.

Definition 3.1 (Regular Deformation). Suppose Σ is an $(n-1)$ -dimensional (abstract) pure simplicial complex and $\Delta(t) = (\Sigma, \beta(t))$, $0 \leq t \leq 1$, is a family of complete non-singular topological fans of dimension n which depend on the parameter t smoothly. Then we call $\Delta(t)$ a *regular deformation* of $\Delta(0)$. Let

$$\beta_i(t) = (b_i(t) + \sqrt{-1}c_i(t), v_i(t)) \in \mathbb{C}^n \times \mathbb{Z}^n, \quad i \in \Sigma^{(1)}. \quad (12)$$

It is clear that $v_i(t) = v_i(0) \in \mathbb{Z}^n$ for each $i \in \Sigma^{(1)}$ since \mathbb{Z}^n is a discrete set. So each $v_i(t)$ is independent on the parameter t . Hence for any complete non-singular topological fan $\Delta = (\Sigma, \beta)$, there always exists some other complete non-singular topological fan $\Delta' = (\Sigma, \beta')$ so that Δ can not be turned into Δ' via any regular deformation.

The following lemma is the key to the main results of this paper.

Lemma 3.2. *Suppose $\Delta(t) = (\Sigma, \beta(t))$, $0 \leq t \leq 1$ is a regular deformation of complete non-singular topological fans of dimension n . Then there exists a diffeomorphism $\psi : X(\Delta(0)) \rightarrow X(\Delta(1))$ with the following properties.*

- (i) ψ is equivariant with respect to the canonical $(S^1)^n$ -action on $X(\Delta(0))$ and $X(\Delta(1))$.
- (ii) ψ maps each open subset $V_I(0) = U(I) \times \{0\} / \text{Ker}(\lambda_{\beta(0)})$ in $X(\Delta(0))$ to $V_I(1) = U(I) \times \{1\} / \text{Ker}(\lambda_{\beta(1)})$ in $X(\Delta(1))$ for any $I \in \Sigma^{(n)}$.

Proof. By definition, $X(\Delta(t)) = U(\Sigma)/\text{Ker}(\lambda_{\beta(t)})$, $0 \leq t \leq 1$. Let $m = |\Sigma^{(1)}|$ be the number of 0-simplices in Σ . Then we have a smooth map Λ defined by

$$\begin{aligned} \Lambda : (\mathbb{C}^*)^m \times [0, 1] &\longrightarrow (\mathbb{C}^*)^n \\ (g, t) &\longmapsto \lambda_{\beta(t)}(g) \end{aligned}$$

$$\text{Let } \Lambda_t^{-1}(e) := \Lambda^{-1}(e) \cap ((\mathbb{C}^*)^m \times \{t\}), \quad t \in [0, 1],$$

where e is the unit element of $(\mathbb{C}^*)^n$. Obviously, $\Lambda_t^{-1}(e) \cong \text{Ker}(\lambda_{\beta(t)})$, $t \in [0, 1]$.

Similarly, for $(S^1)^m \times [0, 1] \subset (\mathbb{C}^*)^m \times [0, 1]$, we have a smooth map

$$\begin{aligned} \widehat{\Lambda} : (S^1)^m \times [0, 1] &\longrightarrow (S^1)^n \subset (\mathbb{C}^*)^n \\ (g, t) &\longmapsto \widehat{\lambda}_{\beta(t)}(g) \quad (\text{see (5)}) \end{aligned}$$

$$\text{Let } \widehat{\Lambda}_t^{-1}(e) := \widehat{\Lambda}^{-1}(e) \cap ((S^1)^m \times \{t\}) \cong \text{Ker}(\widehat{\lambda}_{\beta(t)}), \quad t \in [0, 1].$$

Suppose $\beta_i(t) = (b_i(t) + \sqrt{-1}c_i(t), v_i) \in \mathbb{C}^n \times \mathbb{Z}^n$, $i \in [m] = \Sigma^{(1)}$, where $v_i \in \mathbb{Z}^n$ is a fixed vector for each $i \in [m]$. Then since $\widehat{\lambda}_{\beta(t)}$ is determined only by $\{v_i\}_{i \in [m]}$, so $\widehat{\lambda}_{\beta(t)}$ is independent on t . Hence we have

$$\widehat{\Lambda}^{-1}(e) = \widehat{\Lambda}_0^{-1}(e) \times [0, 1] \subset (S^1)^m \times [0, 1]. \quad (13)$$

Next, we introduce an equivalence relation between any two points (x, t) and (x', t') in $\mathbb{C}^m \times [0, 1]$ by

$$(x, t) \sim (x', t') \iff t' = t, \quad x' = g \cdot x \text{ for some } g \in \Lambda_t^{-1}(e)$$

Let the quotient space of $U(\Sigma) \times [0, 1]$ with respect to \sim be denoted by

$$U(\Sigma) \times [0, 1] / \Lambda^{-1}(e).$$

We use $[(x, t)]$ to denote the equivalence class of (x, t) in $U(\Sigma) \times [0, 1] / \Lambda^{-1}(e)$. We remark that $\Lambda^{-1}(e) \subset (\mathbb{C}^*)^m \times [0, 1]$ is not a group although each $\Lambda_t^{-1}(e)$ is.

Claim: $U(\Sigma) \times [0, 1] / \Lambda^{-1}(e)$ is a smooth manifold (with boundary).

First, we need to show that $U(\Sigma) \times [0, 1] / \Lambda^{-1}(e)$ with the quotient topology is Hausdorff. Let $[(x, t)]$ and $[(x', t')]$ be two different points in $U(\Sigma) \times [0, 1] / \Lambda^{-1}(e)$.

- (a) if $t \neq t'$, then there exists a small real number $\varepsilon > 0$ so that $(t - \varepsilon, t + \varepsilon)$ and $(t' - \varepsilon, t' + \varepsilon)$ are disjoint subintervals of $[0, 1]$. Then $[(x, t)]$ and $[(x', t')]$ are contained in disjoint open subsets $U(\Sigma) \times (t - \varepsilon, t + \varepsilon) / \Lambda^{-1}(e)$ and $U(\Sigma) \times (t' - \varepsilon, t' + \varepsilon) / \Lambda^{-1}(e)$, respectively.
- (b) if $t = t'$ and $x \neq x'$, then $[(x, t)]$ and $[(x', t')] = [x', t]$ are both contained in $U(\Sigma) \times \{t\} / \Lambda^{-1}(e) = U(\Sigma) \times \{t\} / \Lambda_t^{-1}(e) \cong U(\Sigma) / \text{Ker}(\lambda_{\beta(t)}) = X(\Delta(t))$. Since $X(\Delta(t))$ is Hausdorff (see Lemma 6.1 in [1]), so there exist two disjoint open subsets W and W' of $U(\Sigma)$ with $[(x, t)] \in W \times \{t\} / \Lambda_t^{-1}(e)$ and $[(x', t)] \in W' \times \{t\} / \Lambda_t^{-1}(e)$. Then $[(x, t)]$ and $[(x', t')]$ are contained in disjoint open subsets $W \times [0, 1] / \Lambda^{-1}(e)$ and $W' \times [0, 1] / \Lambda^{-1}(e)$, respectively.

So in any case, $[(x, t)]$ and $[(x', t')]$ can be separated by disjoint open subsets of $U(\Sigma) \times [0, 1] / \Lambda^{-1}(e)$. This means that $U(\Sigma) \times [0, 1] / \Lambda^{-1}(e)$ is Hausdorff.

Second, we need to define a smooth structure on $U(\Sigma) \times [0, 1] / \Lambda^{-1}(e)$. Notice that we can cover $U(\Sigma) \times [0, 1] / \Lambda^{-1}(e)$ by finitely many open subsets

$$\{U(I) \times [0, 1] / \Lambda^{-1}(e); I \in \Sigma^{(n)}\}.$$

For any $I \in \Sigma^{(n)}$, let \mathbb{C}^I be the affine space \mathbb{C}^n with coordinates indexed by the elements of I . For any $t \in [0, 1]$, let $\{\alpha_i^I(t)\}_{i \in I}$ be the dual set of $\{\beta_i(t)\}_{i \in I}$. Then similar to (5.1) in [1], we define

$$\bar{\varphi}_I : U(I) \times [0, 1]/\Lambda^{-1}(e) \rightarrow \mathbb{C}^I \times [0, 1] \quad (14)$$

$$\text{by } \bar{\varphi}_I([(z_1, \dots, z_m, t)]) = \left(\left(\prod_{k=1}^m z_k^{\langle \alpha_i^I(t), \beta_k(t) \rangle} \right)_{i \in I}, t \right) = \left((z_i \prod_{k \notin I} z_k^{\langle \alpha_i^I(t), \beta_k(t) \rangle})_{i \in I}, t \right)$$

where $(z_1, \dots, z_m) \in U(I) \subset \mathbb{C}^m$ and $t \in [0, 1]$. The definitions of a dual set and pairing $\langle \cdot, \cdot \rangle$ are given in section 2 of [1].

Here since $\beta_i(t)$ varies smoothly with respect to the parameter t for any $i \in [m]$, so does $\alpha_i^I(t)$ for any $i \in I$. Therefore, the map $\bar{\varphi}_I$ is continuous. And by the same argument in section 5 of [1], we can show that $\bar{\varphi}_I$ is a homeomorphism. So we have a finite set of charts $\{\bar{\varphi}_I : U(I) \times [0, 1]/\Lambda^{-1}(e) \rightarrow \mathbb{C}^I \times [0, 1]\}_{I \in \Sigma^{(n)}}$ which cover $U(I) \times [0, 1]/\Lambda^{-1}(e)$. Moreover, we can see from Lemma 5.2 in [1] that the transition function from such a chart indexed by $I \in \Sigma^{(n)}$ to another one indexed by $J \in \Sigma^{(n)}$ is $\bar{\varphi}_J \circ (\bar{\varphi}_I)^{-1} : \mathbb{C}^I \times [0, 1] \rightarrow \mathbb{C}^J \times [0, 1]$ where

$$\bar{\varphi}_J \circ (\bar{\varphi}_I)^{-1}((z_i)_{i \in I}, t) = \left(\left(\prod_{i \in I} z_i^{\langle \alpha_j^J(t), \beta_i(t) \rangle} \right)_{j \in J}, t \right), ((z_i)_{i \in I}, t) \in \mathbb{C}^I \times [0, 1]. \quad (15)$$

This function is smooth since $z_i \neq 0$ for $i \in I \setminus J$ and $\langle \alpha_j^J(t), \beta_i(t) \rangle = \delta_{ji} \mathbf{1}$ for $i \in J$. Therefore, $\{\bar{\varphi}_I : U(I) \times [0, 1]/\Lambda^{-1}(e) \rightarrow \mathbb{C}^I \times [0, 1]\}_{I \in \Sigma^{(n)}}$ determines a smooth structure on $U(\Sigma) \times [0, 1]/\Lambda^{-1}(e)$. So the claim is proved.

Next, we define a map

$$p : U(\Sigma) \times [0, 1]/\Lambda^{-1}(e) \longrightarrow [0, 1] \text{ where } p([(x, t)]) = t. \quad (16)$$

It is clear that p is a smooth map. And for any $t \in [0, 1]$, we have

$$p^{-1}(t) = U(\Sigma) \times \{t\}/\Lambda_t^{-1}(e) \cong U(\Sigma)/\text{Ker}(\lambda_{\beta(t)}) = X(\Delta(t)).$$

Notice that the set of charts $\{\bar{\varphi}_I : U(I) \times [0, 1]/\Lambda^{-1}(e) \rightarrow \mathbb{C}^I \times [0, 1]\}_{I \in \Sigma^{(n)}}$ defined by (14) restricted to each $p^{-1}(t) \cong X(\Delta(t))$ gives us exactly the set of all normal charts of $X(\Delta(t))$. So $p^{-1}(t)$ is an embedding submanifold of $U(\Sigma) \times [0, 1]/\Lambda^{-1}(e)$.

Let $(S^1)^m \subset (\mathbb{C}^*)^m$ act on $U(\Sigma) \times [0, 1] \subset \mathbb{C}^m \times [0, 1]$ by the natural action

$$g \cdot (x, t) = (g \cdot x, t), \quad \forall g \in (S^1)^m, \forall (x, t) \in \mathbb{C}^m \times [0, 1].$$

Then by (13), we have a smooth action of $(S^1)^m/\widehat{\Lambda}_0^{-1}(e) \cong (S^1)^n$ on the whole $U(\Sigma) \times [0, 1]/\Lambda^{-1}(e)$, whose restriction to each fiber $p^{-1}(t) \cong X(\Delta(t))$ is exactly the canonical $(S^1)^n$ -action on $X(\Delta(t))$.

Obviously, p is a smooth proper submersion. So Ehresmann's fibration theorem (see [6] or [7]) implies that $p : U(\Sigma) \times [0, 1]/\Lambda^{-1}(e) \rightarrow [0, 1]$ is a locally trivial fiber bundle. Therefore, there exists a diffeomorphism from $p^{-1}(0) = X(\Delta(0))$ to $p^{-1}(1) = X(\Delta(1))$. But if we want to require the diffeomorphism to be equivariant with respect to the canonical $(S^1)^n$ -action on $X(\Delta(0))$ and $X(\Delta(1))$, we need to refine the proof of Ehresmann's fibration theorem as follows.

By our previous discussion, $\{U(I) \times [0, 1]/\Lambda^{-1}(e); I \in \Sigma^{(n)}\}$ is a finite open cover of $U(\Sigma) \times [0, 1]/\Lambda^{-1}(e)$ where each $U(I) \times [0, 1]/\Lambda^{-1}(e)$ is an invariant open set under the $(S^1)^n$ -action. Then we can take a $(S^1)^n$ -invariant partition of unity

$$\{f_I : U(I) \times [0, 1]/\Lambda^{-1}(e) \rightarrow \mathbb{R}; I \in \Sigma\}$$

subordinate to this open cover, where each f_I is a $(S^1)^n$ -invariant function.

Next, we take a local vector field Y_I on $U(I) \times [0, 1]/\Lambda^{-1}(e)$ so that

$$p_*(Y_I) = \frac{\partial}{\partial t}.$$

And we set

$$\tilde{Y}_I := \int_{(S^1)^n} Y_I d\mu$$

where $d\mu$ is a Haar measure on $(S^1)^n$. It is clear that

$$p_*(\tilde{Y}_I) = \frac{\partial}{\partial t}.$$

And so we obtain a $(S^1)^n$ -invariant vector field $\tilde{Y} := \sum_{I \in \Sigma} f_I \tilde{Y}_I$ on the whole space $U(\Sigma) \times [0, 1]/\Lambda^{-1}(e)$.

Let ψ_t be the flow of \tilde{Y} on $U(\Sigma) \times [0, 1]/\Lambda^{-1}(e)$. Since \tilde{Y} is $(S^1)^n$ -invariant, ψ_t is also $(S^1)^n$ -invariant. So

$$\psi = \psi_1 : X(\Delta(0)) \cong p^{-1}(0) \longrightarrow p^{-1}(1) \cong X(\Delta(1)).$$

is a $(S^1)^n$ -equivariant diffeomorphism. This proves the property (i) for ψ .

Furthermore, since ψ is equivariant with respect to the $(S^1)^n$ -action, Remark 2.2. implies that ψ will map each characteristic submanifold $X(\Delta(0))_i$ in $X(\Delta(0))$ to the characteristic submanifold $X(\Delta(1))_i$ in $X(\Delta(1))$ for any $i \in \Sigma^{(1)}$. Then by (10), ψ will map the open subset $V_I(0)$ in $X(\Delta(0))$ to $V_I(1)$ in $X(\Delta(1))$ for any $I \in \Sigma^{(n)}$. This finishes our proof. \square

Remark 3.3. For a simplicial complex Σ and two complete non-singular toric fans $\Delta = (\Sigma, \beta)$ and $\Delta' = (\Sigma, \beta')$, it is possible that $X(\Delta)$ is homeomorphic to $X(\Delta')$ while Δ and Δ' can not be connected by any regular deformation. In this situation, it is not so clear whether we can find a diffeomorphism from $X(\Delta)$ to $X(\Delta')$.

Theorem 3.4. *For any ominioriented topological toric manifold X , there always exists a regular deformation of the topological fan of X into a nice topological fan.*

Proof. Suppose X is a $2n$ -dimensional ominioriented topological toric manifold whose topological fan is $\Delta(X) = (\Sigma(X), \beta(X))$, where $\Sigma(X)$ and $\beta(X)$ are defined by (1) and (11), respectively. We will define a sequence of regular deformations of $\Delta(X)$ below. But for the sake of conciseness, we will define each deformation in terms of the deformation on the $b_i(X)$ and $c_i(X)$ components.

Step 1: we deform all $c_i(X)$ to 0 simultaneously by $(1-t)c_i(X)$, $0 \leq t \leq 1$, $i \in [m] := \Sigma^{(1)}(X)$. Obviously, the deformation is regular. We denote the new topological fan obtained from this deformation by $\tilde{\Delta}^{(1)} = (\Sigma(X), \tilde{\beta}^{(1)})$ where

$$\tilde{\beta}_i^{(1)} = (b_i(X) + \sqrt{-1} \cdot 0, v_i(X)) \in \mathbb{R}^n \times \mathbb{Z}^n \subset \mathbb{C}^n \times \mathbb{Z}^n, \quad i \in [m].$$

Step 2: we deform each $b_i(X)$ in $\tilde{\Delta}^{(1)}$ slightly into a vector $b'_i(X)$ with rational coordinates. Notice that the condition on $\{b_i(X), i \in [m]\}$ in the definition of a complete non-singular topological fan is stable under small deformations. So we can choose our deformation of $\tilde{\Delta}^{(1)}$ here to be regular. We denote the new topological fan obtained from this deformation by $\tilde{\Delta}^{(2)} = (\Sigma(X), \tilde{\beta}^{(2)})$ where

$$\tilde{\beta}_i^{(2)} = (b'_i(X) + \sqrt{-1} \cdot 0, v_i(X)) \in \mathbb{Q}^n \times \mathbb{Z}^n \subset \mathbb{C}^n \times \mathbb{Z}^n, \quad i \in [m].$$

Step 3: we can choose a very large positive even integer N so that

$$\widehat{b}_i(X) := N \cdot b'_i(X) \in 2\mathbb{Z}^n \text{ for any } i \in [m].$$

Obviously $\{(1-t)b'_i(X) + t\widehat{b}_i(X), i \in [m]\}$ determines a regular deformation of $\widetilde{\Delta}^{(2)}$. We denote the new topological fan obtained from this deformation by $\widetilde{\Delta}^{(3)} = (\Sigma(X), \widetilde{\beta}^{(3)})$ where

$$\widetilde{\beta}_i^{(3)} = (\widehat{b}_i(X) + \sqrt{-1} \cdot 0, v_i(X)) \in 2\mathbb{Z}^n \times \mathbb{Z}^n \subset \mathbb{C}^n \times \mathbb{Z}^n, \quad i \in [m].$$

Step 4: Let $u_i := \widehat{b}_i(X) - v_i(X) \in \mathbb{Z}^n$ for each $i \in [m]$. Notice in step 3, if we choose a large enough integer N , we can assume $\|v_i(X)\|$ is by far smaller than $\|\widehat{b}_i(X)\|$. Then the distance between the unit vectors $u_i/\|u_i\|$ and $\widehat{b}_i(X)/\|\widehat{b}_i(X)\|$ can be made arbitrarily small. It is easy to see that if $u_i/\|u_i\|$ and $\widehat{b}_i(X)/\|\widehat{b}_i(X)\|$ are very close, $\{(1-t)\widehat{b}_i(X) + tu_i, i \in [m]\}$ will determine a regular deformation of $\widetilde{\Delta}^{(3)}$. We denote the new topological fan obtained from this deformation by $\widetilde{\Delta}^{(4)} = (\Sigma(X), \widetilde{\beta}^{(4)})$ where

$$\widetilde{\beta}_i^{(4)} = (u_i + \sqrt{-1} \cdot 0, v_i(X)) \in \mathbb{Z}^n \times \mathbb{Z}^n \subset \mathbb{C}^n \times \mathbb{Z}^n, \quad i \in [m].$$

For any $i \in [m]$, we have $u_i - v_i(X) = \widehat{b}_i(X) - 2v_i(X) \in 2\mathbb{Z}^n$, so

$$u_i \equiv v_i(X) \pmod{2}.$$

This means that the topological fan $\widetilde{\Delta}^{(4)}$ is nice. Note that the integral vector u_i is not necessarily primitive. By combining the above four steps, we then obtain a regular deformation from $\Delta(X)$ to a nice topological fan $\widetilde{\Delta}^{(4)}$. \square

From Lemma 3.2 and Theorem 3.4, we immediately get the following.

Theorem 3.5. *For any ominioriented topological toric manifold X of dimension $2n$, there exists a $(S^1)^n$ -equivariant diffeomorphism from X to an ominioriented topological toric manifold \widetilde{X} whose topological fan is nice.*

Moreover, we have the following corollary on the transition functions of a topological toric manifold.

Corollary 3.6. *Any topological toric manifold X of dimension $2n$ can be covered by finitely many open charts $\{\phi_j : V_j \rightarrow \mathbb{C}^n\}_{1 \leq j \leq r}$ so that each V_j is a $(\mathbb{C}^*)^n$ -invariant open subset of X and, all the transition functions between these charts are Laurent monomials of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$.*

Proof. By choosing an ominiorientation on X , we can assume $X = X(\Delta)$ where $\Delta = (\Sigma, \beta)$ is a complete non-singular topological fan. Then by Theorem 3.4, there exists a diffeomorphism $\psi : X \rightarrow \widetilde{X} = X(\widetilde{\Delta})$ where $\widetilde{\Delta} = (\Sigma, \widetilde{\beta})$ is a nice topological fan. Moreover, for any $I \in \Sigma^{(n)}$, let $\varphi_I : V_I \rightarrow \mathbb{C}^n$ and $\widetilde{\varphi}_I : \widetilde{V}_I \rightarrow \mathbb{C}^n$ be the normal charts of X and \widetilde{X} , respectively. Then by Lemma 3.2, we have

$$\psi(V_I) = \widetilde{V}_I.$$

Since $(\Sigma, \widetilde{\beta})$ is a nice topological fan, the transition functions between the charts $\{\widetilde{\varphi}_I : \widetilde{V}_I \rightarrow \mathbb{C}^n, I \in \Sigma^{(n)}\}$ are all Laurent monomials of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$. Then $\{\widetilde{\varphi}_I \circ \psi : V_I = \psi^{-1}(\widetilde{V}_I) \rightarrow \mathbb{C}^n, I \in \Sigma^{(n)}\}$ are finitely many open charts which cover X and clearly satisfy all our requirements. \square

Remark 3.7. For an open chart $\widetilde{\varphi}_I \circ \psi : V_I = \psi^{-1}(\widetilde{V}_I) \rightarrow \mathbb{C}^n$ on X in the proof of Corollary 3.6, the $(\mathbb{C}^*)^n$ -action in V_I may not be equivariantly homeomorphic to a smooth linear representation of $(\mathbb{C}^*)^n$ on \mathbb{C}^n via the map $\widetilde{\varphi}_I \circ \psi$, but it does via the map $\varphi_I : V_I \rightarrow \mathbb{C}^n$. The reason is that the diffeomorphism ψ we obtain in Lemma 3.2 is not necessarily $(\mathbb{C}^*)^n$ -equivariant.

There should be some special geometrical properties on a topological toric manifold implied by the existence of the kind of atlas as described in Corollary 3.6. It is interesting to see what these geometric properties are.

In addition, by Theorem 10.2 in [1] and our preceding discussion, we can easily prove the following theorem for quasitoric manifolds.

Theorem 3.8. *Any $2n$ -dimensional quasitoric manifold over a simple convex polytope P^n can be covered by finitely many $(S^1)^n$ -invariant open charts*

$$\phi_j : V_j \rightarrow \mathbb{C}^n, \quad 1 \leq j \leq r \quad \text{where } r = \text{the number of vertices of } P^n,$$

whose transition functions are all Laurent monomials of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$.

Proof. For any $2n$ -dimensional quasitoric manifold M^{2n} over a simple convex polytope P^n , there exists a topological toric manifold $X(\Delta)$ so that M^{2n} is equivariantly homeomorphic to $X(\Delta)$ as $(S^1)^n$ -manifold. In the topological fan $\Delta = (\Sigma, \beta)$, the simplicial complex Σ is the boundary of the dual simplicial polytope of P^n . So the number of $(n-1)$ -simplices in Σ equals the number of vertices of P^n . Moreover, we can require the topological fan Δ is nice by Theorem 3.5. Let ψ be an $(S^1)^n$ -equivariant homeomorphism from M^{2n} to $X(\Delta)$. Then we can use ψ to pull back the atlas on $X(\Delta)$ described in Corollary 3.6 to an atlas on M^{2n} , which will satisfy all our requirements. \square

Remark 3.9. For a $2n$ -dimensional quasitoric manifold M^{2n} , the set of $(S^1)^n$ -invariant open charts described in Theorem 3.8 will determine a smooth structure on M^{2n} . But it is not clear whether this smooth structure should agree with the original smooth structure of M^{2n} (see the remark after the Theorem 10.2 in [1]).

4. Nice topological toric manifolds and real algebraic representation of $(\mathbb{C}^*)^n$

Definition 4.1 (Nice Topological Toric Manifold). A topological toric manifold X is called *nice* if there exists an orientation on X so that the associated topological fan is nice.

Definition 4.2 (Real Algebraic Linear Representation). A faithful smooth linear representation ρ of $(\mathbb{C}^*)^n$ on \mathbb{C}^n is called *real algebraic* if

$$\rho(z_1, \dots, z_n) = (h_1(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n), \dots, h_n(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)) \in (\mathbb{C}^*)^n$$

where $h_j(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$ is a Laurent monomial of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ for each $1 \leq j \leq n$.

By the discussion in section 2 of [1], any faithful smooth complex n -dimensional representation V of $(\mathbb{C}^*)^n$ can be written as

$$V = V\left(\bigoplus_{i=1}^n \chi^{\alpha_i}\right), \quad \text{where } \chi^{\alpha_i} \in \text{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*).$$

We can identify $\text{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*)$ with the row vector space of $\mathbb{C}^n \times \mathbb{Z}^n$ and write

$$\alpha_i = (x_i + \sqrt{-1}y_i, u_i) \in \mathbb{C}^n \times \mathbb{Z}^n$$

where $x_i, y_i \in \mathbb{R}^n$ and $u_i \in \mathbb{Z}^n$ are row vectors. By Lemma 2.1 in [1], it is easy to see the following.

$$V\left(\bigoplus_{i=1}^n \chi^{\alpha_i}\right) \text{ is real algebraic} \iff y_i = 0 \text{ and } x_i \in \mathbb{Z}^n \text{ with } x_i \equiv u_i \pmod{2} \text{ for all } 1 \leq i \leq n$$

Suppose $\{\beta_i\}_{1 \leq i \leq n}$ is a *dual set* of $\{\alpha_i\}_{1 \leq i \leq n}$, i.e. $\lambda_{\beta_i} \in \text{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ so that

$$\lambda_{\beta_j} \circ \chi^{\alpha_i} = \delta_{ij} \text{id}_{(\mathbb{C}^*)^n}, \quad \chi^{\alpha_i} \circ \lambda_{\beta_j} = \delta_{ij} \text{id}_{\mathbb{C}^*} \text{ for all } 1 \leq i, j \leq n.$$

If we write $\beta_i = (b_i + \sqrt{-1}c_i, v_i) \in \mathbb{C}^n \times \mathbb{Z}^n$, it is easy to see the following.

$$V\left(\bigoplus_{i=1}^n \chi^{\alpha_i}\right) \text{ is real algebraic} \iff c_i = 0 \text{ and } b_i \in \mathbb{Z}^n \text{ with } b_i \equiv v_i \pmod{2} \text{ for all } 1 \leq i \leq n$$

Then by our discussion at the end of section 2, for any nice topological fan $\Delta = (\Sigma, \beta)$ of dimension n , the corresponding nice topological toric manifold $X(\Delta)$ can be covered by finitely many $(\mathbb{C}^*)^n$ -invariant open charts $\{\varphi_I : V_I \rightarrow \mathbb{C}^n\}_{I \in \Sigma(n)}$ so that for each $\varphi_I : V_I \rightarrow \mathbb{C}^n$, the $(\mathbb{C}^*)^n$ -action in V_I is equivariant homeomorphic to a real algebraic linear representation of $(\mathbb{C}^*)^n$ in \mathbb{C}^n via the map φ_I .

Similar to the description of toric manifolds in Theorem 2.3, we have the following description of nice topological toric manifolds in terms of transition functions of charts up to (weakly) equivariant diffeomorphism.

Theorem 4.3. *Suppose M^{2n} is a $2n$ -dimensional closed smooth manifold with an effective smooth action of $(\mathbb{C}^*)^n$ having an open dense orbit. Then M^{2n} is (weakly) equivariantly diffeomorphic to a nice topological toric manifold if and only if M^{2n} can be covered by finitely many $(\mathbb{C}^*)^n$ -invariant open charts $\{\phi_j : V_j \rightarrow \mathbb{C}^n\}_{1 \leq j \leq r}$ so that all the transition functions between these charts are Laurent monomials of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ and, for at least one chart $\phi_j : V_j \rightarrow \mathbb{C}^n$, the $(\mathbb{C}^*)^n$ -action in V_j is equivariant homeomorphic to a real algebraic linear representation of $(\mathbb{C}^*)^n$ on \mathbb{C}^n via the map ϕ_j .*

Proof. Suppose there exists a (weakly) equivariant diffeomorphism ψ from M^{2n} to a nice topological toric manifold $X = X(\Delta)$ where $\Delta = (\Sigma, \beta)$ is a nice topological fan. By our previous discussion, $X(\Delta)$ can be covered by finitely many $(\mathbb{C}^*)^n$ -invariant open charts $\{\varphi_I : V_I \rightarrow \mathbb{C}^n\}_{I \in \Sigma(n)}$ so that the $(\mathbb{C}^*)^n$ -action in each V_I is equivariant homeomorphic to a real algebraic linear representation of $(\mathbb{C}^*)^n$ in \mathbb{C}^n via the map φ_I . It is clear that each $\psi^{-1}(V_I)$ is a $(\mathbb{C}^*)^n$ -invariant open subset of M^{2n} . So we can cover M^{2n} by finitely many $(\mathbb{C}^*)^n$ -invariant open charts

$$\{\phi_I \circ \psi : \psi^{-1}(V_I) \rightarrow \mathbb{C}^n\}_{I \in \Sigma(n)}$$

whose transition functions are all Laurent monomials of $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$. Moreover, since ψ is a (weakly) equivariant diffeomorphism, so the $(\mathbb{C}^*)^n$ -action in each $\psi^{-1}(V_I) \subset M^{2n}$ is (weakly) equivariantly diffeomorphic to a real algebraic linear representation of $(\mathbb{C}^*)^n$ in \mathbb{C}^n via $\phi_I \circ \psi$. This proves the “only if” part of the theorem.

The “if” part of the theorem is very similar to the proof of Theorem 2.3, so we leave it as an exercise to the reader. \square

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АННОТАЦИЯ

Мы показываем, что любое топологическое торическое многообразие может быть покрыто конечным числом открытых карт так, что функции перехода между этими картами являются мономами Лорана от z_j и \bar{z}_j . Мы также описываем торические многообразия и некоторый специальный класс топологических торических многообразий с точностью до (слабо) эквивариантного диффеоморфизма в терминах функций перехода между картами.

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Ключевые слова: *торическое многообразие, квазиторическое многообразие, моном Лорана.*